ON MINIMAL INVARIANCE FOR FIXED POINT PROPERTY

UDO-UTUN, X. A.

Department of Mathematics and Statistics
University of Uyo, Uyo – Nigeria
xavierudoutun@gmail.com

ISSN: 2141 – 3290
www.wojast.com

ABSTRACT

We have shown how to prove fixed point property for non expansive mappings in the absence of normal structure using minimal invariance and methods of almost contractions. Our result is significant in combining the geometric minimal invariance with metrical properties of weak contractions to obtain purely metrical approach to fixed point property.

MSC: 47H09, 47H10, 54H25;

INTRODUCTION

We recall that a metric space X is said to have a fixed point property for a class of mappings $T : X \to X$ if it's fixed point set *Fix*(*T*) is not empty (i.e if $Fix(T) \neq \emptyset$;) for each mapping T in the class. The contraction mapping principle asserts that a complete metric space has the fixed point property for all contraction mappings and implies that a bounded closed subset of a Banach space has the fixed point property for all contractions.

Definitions of Terms and Notions

For the purpose of coherent interaction and reference purposes we use this subsection to present basic terms and notions. Below $C \subset E$ denotes a bounded closed convex subset of a Banach space E, $T : C \to E$ denotes a mapping of C into E, $K \subset C$ denotes a bounded closed convex subset of C then the diameter of a set K denoted by *diam*(K) is given by *diam*(K) = $\sup\{\| x - y \| : x, y \in K\}$:

Diametral Sets: A convex subset $K \subset C$ is said to be *diametral* if:

$$\sup_{y \in K} \| x - y \| = diam(K) \text{ for any } x \in K.$$

Normal structure: Sets which do not contain diametral convex subsets other than singletons are said to have normal structure.

Lipschitzian Condition: A mapping $T : C \to C$ is called *lipschitzian* if there exists a constant $L > 0$ (called *Lipschitz constant*) such that T satisfies the *Lipschitz condition.* $\| Tx - Ty \| \leq L \| x - y \|$. The mapping T is said to be *non-expansive* if $L = 1$ and T is called a *contraction mapping* when $L < 1$. Important studies of fixed points of non-expansive mappings include Kirk (1969), Liu and Kang (2001) and Wisnicki (2012). In addition, T is called *contractive* if $\|Tx - Ty\| < \|x - y\|$.It should be observed here that contractive mappings define the class of mappings that lie in between the class of contraction mappings and the class of non-expansive mappings. That is, all contractions are included in the class of contractive mappings but not all contractive mappings are contractions and both classes of contractive mappings and contractions are subclasses of the class of non-expansive mappings. For more readings on contractive mappings we have made reference to Belluce and Kirk (1969), Berinde (2007), Berinde and Pacurar (2016) and Ciric (1975).

Fixed Point Property: A mapping $T : C \to C$is said to possess a *fixed point* if there exists a point $x \in C$such that $x^* = Tx^*$ and the collection of all fixed points of the mapping T is denoted by $Fix(T)$. A a set Cissaid to have the *fixed point property* for a class of mappings $T : C \to C$if$Fix(T) \neq \emptyset$; for all the mappings in that class.

Approximate Fixed Point Sequence: A sequence $(x_n)_{n\geq 1} \subset C$is called *approximate fixed point sequence* for a mapping T if $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$.

Minimal Invariants: Let $T : C \to C$be a mapping then $K \subset C$is called T-minimal invariant if K is the smallest set such that $T : K \to K$i.e if there exists a set $K' \subset C$such that$T : K' \to K'$, then $K \subset K'$.

The drastic departure of behaviors of contractive mappings and non-expansive mappings with reference to contraction mapping principle led to the rigorous development of the theory of non-expansive mappings which originated in 1965when Browder and Gohde (1965) proved that each nonempty bounded and convex subset of a uniformly convex Banach space has the fixed point property for non-expansive mappings. The requirement of uniform convexity of the Banach was to avoid the presence of certain *diametral sets* which contain the undesirable nontrivial minimal invariant sets. So, the foundation of this theory is also credited to Kirk (1965) who modified the requirement of uniform convexity of Banach space, the same year, by establishing that weak compactness of convex subsets of any Banach space with normal structure was sufficient for the fixed point property. We recall that convex sets are said to possess normal structure if they do not contain diametral sets. However, there are (weakly compact) diametral sets which are no minimal invariants for any non-expansive mapping Aksoy and Khamsi (1990) and Goebel (1975). Such diametral sets have been shown to have the WFPP for non-expansive mappings, provided they contain almost non-diametral points (Goebel, 1975, Goebel and Kirk, 1990; Karlovitz, 1976).

Theorem 1. (Goebel 1975)
If any convex diametral subset K (not a singleton) of a weakly compact convex set C contains an almost non-diametral point, then C has FPP for nonexpansive mappings.

But there still remains an unsolved problem in this direction concerned with whether there really exist a Banach space with diametral convex subset which contains an almost non-diametral element. This leads us to the important question of if there is an alternative approach to investigation FPP of diametral sets for non-expansive mappings; in particular, if there are some unexplored features of contraction mappings which can aid investigation of fixed points of non-expansive mappings other than the lipschitzian displacement condition $\| Tx - Ty \| \leq k \| x - y \|$, $k \in (0,1)$. If such unexplored features exist then we may investigate fixed point property of diametral sets (which are not minimalinvariants) since they have fixed point property for contractions.

The aim of this work is to formulate condition for fixed point property of a bounded closed convex subset (inclusive of diametral sets) of Banach space for non-expansive mappings which does not depend on the notion of almost non-diametral points. An important and outstanding feature of contraction mappings considered in this work is that the convex hull$\overline{cov}(x_n)$of an approximate fixed point sequence for a contraction mapping is not a minimal invariant. So, a non-expansive mapping is not expected to have a fixed point if the closed convex hull $\overline{cov}(x_n)$ of its approximate fixed point sequence (x_n) is a minimal invariant subset. So a natural problem emerges concerning condition under whichthe closed convex hull $\overline{cov}(x_n)$ is not minimal T-invariant or otherwise. In this presentation we prove the fixed point property of bounded closed convex subsets $C \subset E$ of Banach spaces E for non-expansive mappings by formulating condition under which the closed convex hull $\overline{cov}(x_n)$ of their approximate fixed point sequences $(x_n)$are not a minimal invariants. The most fundamental of findings, in this direction, is, probably, the Karlovitz-Goebel lemma (Goebel, 1975 and Karlovitz, 1976) stated below:

Lemma 2. *(*Goebel (1975) – Karlovitz (1976)) *Let K be a minimal set for T. Then for any approximate fixed point sequence* (x_n)*in K, wehave*

$$\lim_{n\to\infty} \| x_n - x \| = diam(K) \text{ for all} x \in K. \quad (1)$$

The *almost-contraction* fixed point principle used in this work due to Berinde and Pacurar (2016) is stated below:

Definition 3. *(*Berinde (2003), Berinde (2007), Berinde and Pacurar (2016))
Let (*X, d*) *be a metric space,*$\delta \in (0,1)$*and* $k \geq 0$*, then a mapping* $T : X \to X$ *is calledalmost contraction (formerly,* (δ, k)*–weak contraction or a weak contraction) if and only if*

$$d(Tx,Ty) \leq \delta d(x,y) + kd(y,Tx), for\ all\ x,y \in X. \quad (2)$$

Theorem 4. *(*Berinde, 2003, Berinde, 2007, Berinde and Pacurar, 2016)
Let (*X, d*) *be a complete metric space and* $T : X \to X$ *be a* (δ, k)*–weak contraction (i.e almost contraction). Then:*
(i) $ix(T) = \{x \in X : Tx = x\} \neq \emptyset$;.
(ii) For any $x_0 \in X$ *the Picard iteration* $\{x_n\}$ *given by* $x_{n+1} = T^{n+1}x_0, n = 0,1,2,\ldots$ *converges to some*$x^* \in Fix(T)$.
(iii) The following estimates

$$d(x_n,x^*) \leq \frac{\delta^n}{1-\delta} d(x_0,x_1), n = 0,1,2,,\ldots$$

$$d(x_n,x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1},x_n), n = 0,1,2,,\ldots$$

hold, where δ *is the constant appearing in (2).*
(iv) Under the additional condition that there exists $\theta \in (0,1)$*and some* $k_1 \geq 0$*such that*

$$d(Tx,Ty) \leq \theta d(x,y) + k_1 d(y,Tx)\ \ for\ all\ x,y \in X. \quad (3)$$

then the fixed point x^**is unique and the Picard iteration converges at the rate:*

$$d(x_n,x^*) \leq \theta d(x_{n-1},x^*), \qquad n \in \mathbb{N}.$$

Theorem 5 (Berinde and Pacurar, 2008), Berinde and Pacurar, 2016)
Let (*X, d*) *be a complete metric space and* $T : X \to X$ *be an almost contraction.*
Then T is continuous at p for any $p \in Fix(T)$.

It has been well established (Berinde, 2003, Berinde, 2007, Berinde and Pacurar, 2016) that the class of almost contractions includes so many classes of weak contractions like Banach contractions and the Zamfirescu operators.

Definition 6. (Wisnicki2012)
Let $T : X \to X$ *be a mapping on a metric space* (*X, d*)*, then:*
T is called asymptotically regular if it satisfies $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$.

In this presentation we desire to combine minimal invariant technique and almost contraction technique to investigate fixed point property of a bounded closed convex subset C of a Banach space E for non-expansive mappings$T : C \to C$using the hypothesis:

(H) There exist an approximate fixed point sequence $(x_n)_{n\geq 1} \sqsubset C$of T and a constant $\eta \geq 1$such that

$$\| x_n - Tx_m \| \leq \eta \| x_m - x_n \|, m < n. \quad (4)$$

MAIN RESULTS

First, we prove that contraction mappings satisfy the hypothesis (H). Lemma 7 below is useful in the sequel:

Lemma 7. *Let* $(x_n)_{n\geq 1}$*be an approximate fixed point sequence, for a non-expansive mapping* $T : C \to C$ *of a convexset C, defined by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n. \quad (5)$$

If $\lambda \in \left(0, \frac{1}{2}\right)$ *then:*

$$\| x_n - x_{n+1} \| \leq \| x_n - x_m \| \text{ for all } m < n.$$

Proof

We recall that the Krasnoselskii averaged scheme $x_{n+1} = (1-\lambda)x_n + \lambda T x_n$ satisfies these two estimates: (a) $\| x_n - x_{n+1} \| = \lambda \| x_n - Tx_n \|$ and $\| x_{n+1} = Tx_n \| = (1-\lambda) \| x_n - Tx_n \|$;

so, using $\lambda < \frac{1}{2}$, we have (b) $\| x_n - x_{n+1} \| < \| x_{n+1} - Tx_n \|$ and (c) $\| x_{n+k} - Tx_n \| < \| x_n - Tx_n \|$; $k = 1, 2, \ldots$. Next, assuming on the contrary that $\| x_n - x_{n+2} \| < \| x_n - x_{n+1} \|$ for some $n \in \mathbb{N}$ and using (a), (b) and (c) in what follows, we obtain

$$\begin{aligned} \| x_n - Tx_n \| &\leq \| x_n - x_{n+2} \| + \| x_{n+2} - Tx_n \| \\ &< \| x_n - x_{n+1} \| + \| x_{n+1} - Tx_n \| \text{ (using (c))} \\ &= \lambda \| x_n - Tx_n \| + (1-\lambda) \| x_n - Tx_n \| \text{ (using (a))}. \end{aligned}$$

$$\Longrightarrow \| x_n - Tx_n \| < \| x_n - Tx_n \|.$$

A contradiction, completing the proof. ■

Similarly, we can show that for contractions $\| x_n - Tx_n \| \leq \| x_n - Tx_m \|$; *m* < *n*.

Proposition 8. *Let C be a bounded closed convex subset of a Banach space E, then every contraction mapping* $T : C \to C$ *satisfies (H), i.e there exist an approximate fixed point sequence* $(x_n) \sqsubset C$ *for T and a constant* $\eta \geq 1$ *such that* $\| x_n - Tx_m \| \leq \eta \| x_n - x_m \|$, *m* < *n*.

Proof

Choosing $\lambda \in \left(0, \frac{1}{2}\right)$ and $\eta \geq 1$ such that $\lambda = \frac{1}{\eta+1}$, it is straight forward showing that the approximate fixed point sequence defined by (5) satisfies (H) for any contraction mapping *T*:

$$\| x_n - Tx_m \| \leq \| x_{m+1} - Tx_m \| \leq \eta \| x_n - x_m \|. \blacksquare$$

Proposition 9. *Let* $(x_n) \sqsubset C$ *be approximate fixed point sequence for a nonexpansive mapping* $T : C \to C$. *If C contains a minimal invariant set K for the mapping T such that diam(K) >0 and* $(x_n) \sqsubset K$. *Then K =* $\overline{cov}(x_n)$.

Proof

If *K* is a singleton, the proof is self-evident. So, we assume *K* is not a singleton and let $(x_n) \sqsubset K$ be an approximate fixed point sequence for *T*. Since *K* is closed and convex it follows that $\overline{cov}(x_n) \subseteq K$. To complete the proof, it suffices to prove the converse $K \subseteq \overline{cov}(x_n)$. Our approach exploits the Karlovitz-Goebel lemma and the non-expansiveness of *T* i.e $\| Tx_m - Tx_n \| \leq \| x_n - Tx_m \|$ which implies that $\overline{cov}(T(x_n)) \subseteq \overline{cov}(x_n)$ and $\| x_n - Tx_m \| \leq \| x_n - Tx_m \| + \| x_n - Tx_n \|$.

Since *K* is *T* invariant it follows that for any $m, n \in \mathbb{N}$, $Tx_m, Tx_n \in K$ given that $x_m, x_n \in K$, yielding $\lim_{n\to\infty} \| x_n - Tx_m \| \leq \lim_{n\to\infty} \| x_n - x_m \| \leq diam(K)$. On application of Lemma 2 (i.e Karlovitz-Goebel lemma), we have $diam(K) = \lim_{n\to\infty} \| x_n - Tx_m \| \leq \lim_{n\to\infty} \| x_n - x_m \| \leq diam(K)$. Hence $K \subseteq \overline{cov}(x_n)$, completing the proof. ■

Remark 10. With Karlovitz-Goebel lemma, we can prove Proposition 9 (i.e *K* = $\overline{cov}(x_n)$) in just one line viz: $diam(K) = \lim_{n\to\infty} \| x_n - Tx_m \| \leq \lim_{n\to\infty} \| x_n - x_m \| \leq diam(K)$

.

Theorem 11. *Let C be a weakly compact closed convex subset of a Banach space E, T :* $T : C \to C$ *be a nonexpansive mapping and* $(x_n)_{n\geq 1}$ *an approximate fixed point sequence for T. Then* $ix(T) \neq \emptyset$; *provided T satisfies (H) i.e: There exist an approximate fixed point sequence* $(x_n)_{n\geq 1} \sqsubset C$ *for T and a constant* $\eta \geq 1$ *such that* $\| x_n - Tx_m \| \leq \eta \| x_n -$

$x_m \parallel$, $m < n$.Further, the sequence $(x_n)_{n\geq 1}$defined by $z_{n+1} = (1-\lambda)z_n + \lambda T z_n$ converges to a fixed point of T, where $\lambda = \frac{1}{\eta+1}$.

Proof

Suppose $Fix(T) = \emptyset$ then by (1) in Karlovitz-Goebel (Lemma 2) C contains a diametral set K which is minimal invariant set under T and by Proposition 9 $K = \overline{cov}(x_n)$. Hence, for a fixed $m \in \mathbb{N}$, we have $\lim_{n\to\infty} \parallel x_n - Tx_m \parallel = diam(K) = \lim_{n\to\infty} \parallel x_n - Tx_m \parallel$.We define the Krasnoselskii averaged mapping $S_\lambda$on K by $S_\lambda x = (1-\lambda)x + \lambda Tx$where $\lambda = \frac{1}{\eta+1}$.

It follows from Lemma 1 that if $Fix(T) = \emptyset$, then for a fixed $m \in \mathbb{N}$ we also have$\lim_{n\to\infty} \parallel S_\lambda x_n - S_\lambda x_m \parallel = diam(K)$,since$S_\lambda x_n - S_\lambda x_m = (x_n - x_m) + \lambda(x_m - Tx_m) - \lambda(x_n - Tx_n)$and (x_n) is an approximate fixed point sequence for$T$.We shall obtain a contradiction by showing that $S_\lambda$is an almost contraction mapping on K which has a fixed point in K since $K = \overline{cov}(x_n)$ is closed and convex.

For any $n, m \in \mathbb{N}$ with $m < n$, we have (using$\parallel x_n - S_\lambda x_n \parallel \leq \parallel x_n - S_\lambda x_m \parallel$):

$$\begin{aligned}
\parallel S_\lambda x_n - S_\lambda x_m \parallel &\leq \lambda \parallel x_n - Tx_n \parallel + \parallel x_n - S_\lambda x_m \parallel \leq \parallel S_\lambda x_n - Tx_n \parallel + (1+\lambda)x_n - S_\lambda x_m \parallel \\
&= \lambda(1-\lambda) \parallel x_n - Tx_n \parallel + (1+\lambda) \parallel x_n - S_\lambda x_m \parallel \\
&\leq \lambda(1-\lambda) \parallel x_n - Tx_m \parallel + \lambda(1-\lambda) \parallel Tx_m - Tx_n \parallel + (1+\lambda) \parallel x_n - S_\lambda x_m \parallel \\
&\leq \lambda(1-\lambda)\eta \parallel x_n - x_m \parallel + \lambda(1-\lambda) \parallel x_m - x_n \parallel + (1+\lambda) \parallel x_n - S_\lambda x_m \parallel \\
&\leq \lambda(1-\lambda)(\eta+1) \parallel x_n - x_m \parallel + (1+\lambda) \parallel x_n - S_\lambda x_m \parallel \\
&= (1-\lambda) \parallel x_n - x_m \parallel + (1+\lambda) \parallel x_n - S_\lambda x_m \parallel \left(using\ \lambda = \frac{1}{\eta+1}\right) \qquad (9)
\end{aligned}$$

DISCUSSION AND CONCLUSION

Hence $S_\lambda$is an almost contraction on a subset (x_n) of K i.e $S_\lambda$is a $(\delta, L) - weak$ contraction with $\delta = (1-\lambda)$and$L = (1+\lambda)$. Therefore given $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that$\parallel S_\lambda x_n - S_\lambda x_m \parallel < \varepsilon$ for all $n > m$. This contradicts$Fix(T) = \emptyset$. therefore T has a fixed point and the iteration scheme $z_{n+1} = S_\lambda z_n$converges to a fixed point of T for any $x_n \in (x_n)_{n\geq 1}$. End of proof. ■

REFERENCES

Aksoy, A. G. and Khamsi, M. A. (1990).*Nonstandard Methods in Fixed Point Theory*, Springer, New York, 139p.

Alspach, D. E. (1981).A fixed point free non-expansive map. *Proc. Amer. Math. Soc.,* 82: 423-424.

Belluce, L. P. and Kirk, W. A. (1969).Fixed point theorems for certain classes of non-expansive mappings. *Proc. Amer. Math. Soc.* 20: 141–146.

Berinde, V. (2003).On the approximation of fixed points of weak contractive mappings. *Carpathian J. Math.* 19 (1): 7-22.

Berinde, V. (2007). *Iterative Approximations of Fixed Points*, Springer-Verlag, Berlin - Heidelberg, 322p.

Berinde, V. and Pacurar, M. (2008).Fixed point theorems and continuity of almost contractions. *Fixed Point Theory*, 19 (2): 23 - 34.

Berinde, V. and Pacurar, M. (2016)..*Iterative Approximation of Fixed Points of Single-Valued almost Contractions* Elsevier Inc. pp 1 - 87

Browder, F. E. (1965). Non-expansive nonlinear operators in a Banach space. *Proc. Nat. Acad. Sci.* USA. 54: 1041–1044.

Ciric, L. B. (1975).Fixed and periodic points for a class of contractive operators, *Publication de l'Institut Mathematique* 18(32): 57 - 69.

Goebel, K. (1975).On the structure of the minimal invariant sets for non-expansive mappings. *Annal. Univ. Mariae Curie-Sklodowska*, 29: 73–77.

Goebel, K. and Kirk, W. A. (1990).*Topics in Metric Fixed Point Theory*, Cambridge University Press, New York, 244p.

Ishikawa, S. (1976).Fixed points and iteration of a non expansive mapping in a Banach space. *Proc. Amer. Math. Soc.*, 59: 65-71.

Karlovitz, L. (1976), Existence of fixed points for non-expansive mappings in spaces without normal structure, *Pacific J. Math*. 66: 153–156.

Kirk, W. A. (1965).A fixed point theorem for mappings which do not increase distances. *Amer. Math. Monthly* 72: 1004–1006.

Kirk, W. (1969).On mappings with diminishing orbital diameters. *J. London Math. Soc,* 44: 107-111.

Liu, Z. and Kang, S. M. (2001).On mappings with diminishing orbital diameters, *IJMMS - Hindawi Publishing Corp,* 27 (6): 341–346.

Wisnicki, A. (2012). On the structure of fixed-point sets of asymptotically regular semi-groups. *J. Math. Anal. Appl*., 393: 177–184.