APPROXIMATE SOLUTIONS OF EXPONENTIAL SCREENED PLUS YUKAWA POTENTIAL WITH NIKIFOROV-UVAROV FUNCTIONAL ANALYSIS METHOD

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ABSTRACT

In this work we applied the Greene-Aldrich approximation to centrifugal term to find an approximate bound state solution of Schrodinger equation with exponential screened plus Yukawa potential with Nikiforov-Uvarov functional analysis method. We obtained the energy-eigen equation and unnormalised wave function expressed in terms of Jacobi polynomial. The proposed potential reduces to exponential screened and Yukawa potential as special cases.

INTRODUCTION

Much realistic potential which describe quantum mechanical systems are not usually exactly solvable in the Schrodinger picture, except for a few occasions such as Harmonic oscillator, Coulomb potential, etc. Therefore, finding exact analytical solution of Schrodinger equation for a given potential corresponding to a physical system of interest constitutes one of the major challenges in quantum mechanics. This is a common problem, often encountered in almost every branch, such as atomic, molecular, solid-state, nuclear, particle and plasma physics, etc. A large number of attractive promising approximate formalisms have been developed ever since the inception of theory, which can provide highly accurate or near- exact results in some cases. These equations are solved by means of different methods for exactly solvable potentials such as Supersymmetry Quantum Mechanics (SUSYQM) (Gereiner,2000; Quense,2013; Yasuk *et al.*, 2005; Xu *et al.*, 2009; Liu *et al.*, 2009 and Chen *et al.*,2009), time-dependent perturbation (Sutherland, 1998), asymptotic iteration method (AIM) (Hou *et al.*,1999;Soylu*et al.*,2007; Soylu *et al.*, 2008 and Hamzavi *et al.*,2002), factorization method (Guo, 2002 and Zhang and Wang, 2005), functional analysis (Jia *et al.*,2006), Nikiforov–Uvarov (NU) method (Qiang*et al.*,2007; Arda*et al.*,2010; Dong and Gu,2008 and Aydogdu and Sever, 2009), and others (Ikhdair, 2010; Hamzavi *et al.*,2010; Nikiforov and Uvarov, 1988; Bayrak *et al.*,2006; Gonul*et al.*, 2000; Egrifes *et al.*, 2000 ,Jia *et al.*,2004 and Jia *et al.*, 2003). Yasuk*etal.* (2005) presented an alternative simple method for the exact solution of the Klein–Gordon equation (KGE) in the presence of non-central equal scalar and vector potentials, by using the Nikiforov–Uvarov method (Nikiforov and Uvarov, 1988). Recently, the study of exponential-type potentials has attracted much attention to many scientists both in non-relativistic quantum mechanics and in relativistic quantum mechanics (Bayrak *et al.*,2006; Gonul *et al.*, 2000; Egrifes *et al.*, 2000; Jia *et al.*, 2004; Jia *et al.*, 2003; Jia *et al.*, 2002; Dong and Garcia-Ravelo, 2007; Zou *et al.*, 2005; Jia *et al.*, 2006; Greene and Aldrich., 1976; Dominguez-Adame and Rodriquez, 1995; Chen*et al.*, 2004; Mehmet and Harun.,2004 and Chetouani*et al.*, 1996). These potentials include the Hulthen potential (Bayrak *et al.*,2006 and Gonul *et al.*,2000), the multi parameter exponential type potentials (Egrifes *et al.*,2000; Jia *et al.*,2004; Jia *et al.*, 2003 and Jia *et al.*,2002), the Manning-Rosen potential (Mehmet *et al.*,2004) and the Eckart-type potential (Zou *et al.*,2005 and Jia *et al.*, 2006). It should be mentioned that most contributions appearing in the literature concern with the s-wave case. However, for the l-wave, one can only solve approximately by using a suitable approximation scheme (Lu, 2005).Based on the motivation derived from the significance of exponential-type potentials, the authors have proposed an exponential screened plus Yukawa potential of the form

$$V(r) = D_0(1 + e^{-2\alpha r}) + \frac{D_1 e^{-\alpha r}}{r} \quad , \tag{1}$$

where D_0 and D_1 represent the coupling strength constant and $D_0 = \frac{D_e}{2} > 0$, where D_e is the dissociation energy while α is the screening parameter. This potential can be used to represent the effective interaction in many-electron atoms; also they have important applications in solid-state, nuclear and plasma physics as well as in field theory (Ferrell and Scalapino, 1974; Hikami and Brezin, 1979; Weisbuch and Vinter,1993; Harrison ,2000 and Shukla and Eliasson.,2008). Lately, the effect of screening on atomic photoionization in H and He^+ has been studied by means of Yukawa potential (Lin *et al*., 2010). Also the ground and excited resonances in two-electron systems such as He, Molecular H_2 in generalized screened potential have been investigated (Ghoshal and Ho, 2009 and Ghoshal and Ho,2011).

METHODS

NU –Functional Analysis (NUFA) Method

Using the concepts of Nikiforov Uvarov (NU) method (Nikiforov and Uvarov, 1988), parametric Nikiforov-Uvarov method (Harrison, 2000) and functional analysis method (Ikot *et al*.,2021). Ikot *et al*. (2021) proposed a simple and elegant method for solving a second-order differential equation of the hypergeometric type called Nikiforov-Uvarov Functional Analysis (NUFA) method. This method is easy and simple just as parametric NU method. Unlike the NU method which involved looking for the square of the polynomials and other conditions which makes it complicated, the NUFA is very easy to use to obtain the energy and the wave function once the wave equations have been properly transformed and the singularities identified. As it is well known, the NU is used to solve a second-order differential equation of the form (Tezcan and Sever, 2009)

$$\psi_n''(s) + \frac{\tilde{\tau}}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}}{\sigma^2(s)}\psi_n(s) = 0, \tag{2}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. (Abramowitz and Stegun, 1972) latter introduced the parametric form of NU method in the form

$$\psi'' + \frac{\alpha_1 - \alpha_2 s}{s(1-\alpha_3 s)}\psi' + \frac{1}{s^2(1-\alpha_3 s)^2}\left[-\xi_1 s^2 + \xi_2 s - \xi_3\right]\psi(s) = 0 \tag{3}$$

where α_i and $\xi_i\ (i = 1,2,3)$ are all parameters. It can be observed in Eq. (2) that the differential equation has two singularities at $s \to 0$ and $s \to \frac{1}{\alpha_3}$, thus we take the wavefunction in the,

$$\psi(s) = s^{\mu}(1-\alpha_3 s)^{\nu} f(s) \tag{4}$$

Substituting Eq. (4) into Eq. (3) leads to the following equation,

$$\begin{aligned} &s(1-\alpha_3 s)f''(s) + \left[\alpha_1 + 2\mu - (2\mu\alpha_3 + 2\nu\alpha_3 + \alpha_2)s\right]f'(s) - \\ &\alpha_3\left(\mu + \nu + \frac{\alpha_2}{\alpha_3} - 1 + \sqrt{\left(\frac{\alpha_2}{\alpha_3} - 1\right)^2 + \frac{\xi_1}{\alpha_3}}\right)\left(\mu + \nu + \frac{\alpha_2}{\alpha_3^2} - 1 + \sqrt{\left(\frac{\alpha_2}{\alpha_3} - 1\right)^2 + \frac{\xi_1}{\alpha_3}}\right) \\ &+\left[\frac{\mu(\mu-1) + \alpha_1\mu - \xi_3}{s} + \frac{\alpha_2\nu - \alpha_1\alpha_3\nu + \nu(\nu-1)\alpha_3 - \frac{\xi_1}{\alpha_3} + \xi_2 - \xi_3\alpha_3}{(1-\alpha_3 s)}\right]f(s) = 0 \end{aligned} \tag{5}$$

Equation (5) can be reduced to a Gauss hypergeometric equation if and only if the following functions vanished,

$$\mu(\mu-1)+\alpha_1\mu-\xi_3=0 \tag{6}$$

$$\alpha_2 v-\alpha_1\alpha_3 v+v(v-1)\alpha_3-\frac{\xi_1}{\alpha_3}+\xi_2-\xi_3\alpha_3=0 \tag{7}$$

Thus, Eq. (5) becomes

$$s(1-\alpha_3 s)f''(s)+\left[\alpha_1+2\mu-(2\mu\alpha_3+2v\alpha_3+\alpha_2)s\right]f'(s)-$$
$$\alpha_3\left(\mu+v+\frac{\alpha_2}{\alpha_3}-1+\sqrt{\left(\frac{\alpha_2}{\alpha_3}-1\right)^2+\frac{\xi_1}{\alpha_3}}\right)\left(\mu+v+\frac{\alpha_2}{\alpha_3^2}-1+\sqrt{\left(\frac{\alpha_2}{\alpha_3}-1\right)^2+\frac{\xi_1}{\alpha_3}}\right)f(s)=0 \tag{8}$$

Solving Eqs. (6) and (11) completely give,

$$\mu=\frac{(1-\alpha_1)\pm\sqrt{(1-\alpha_1)^2+4\xi_3}}{2} \tag{9}$$

$$v=\frac{(\alpha_3+\alpha_1\alpha_3-\alpha_2)\pm\sqrt{(\alpha_3+\alpha_1\alpha_3-\alpha_2)^2+4\left(\frac{\xi_1}{\alpha_3}+\alpha_3\xi_3-\xi_2\right)}}{2} \tag{10}$$

Equation (10) is the hypergeometric equation type of the form [48],

$$x(1-x)f''(x)+\left[c+(a+b+1)x\right]f'(x)-abf(x)=0 \tag{11}$$

Using Eqs. (4), (8) and (11), we obtain the energy equation and the corresponding wave equation, respectively, for the NUFA method as follows:

$$\mu^2+2\mu\left(v+\frac{\alpha_2}{\alpha_3}-1+\frac{n}{\sqrt{\alpha_3}}\right)+\left(v+\frac{\alpha_2}{\alpha_3}-1+\frac{n}{\sqrt{\alpha_3}}\right)^2-\left(\frac{\alpha_2}{\alpha_3}-1\right)^2-\frac{\xi_1}{\alpha_3^2}=0 \tag{12}$$

$$\psi(s)=Ns^{\frac{(1-\alpha_1)+\sqrt{(\alpha_1-1)^2+4\xi_3}}{2}}(1-\alpha_3 s)^{\frac{(\alpha_3+\alpha_1\alpha_3-\alpha_2)+\sqrt{(\alpha_3+\alpha_1\alpha_3)^2+4\left(\frac{\xi_1}{\alpha_3^2}+\alpha_3\xi_3-\xi_2\right)}}{2}}\,{}_2F_1(a,b,c;s) \tag{13}$$

where a, b, c are given as follows,

$$a=\sqrt{\alpha_3}\left(\mu+v+\frac{\alpha_2}{\alpha_3}-1+\sqrt{\left(\frac{\alpha_2}{\alpha_3}-1\right)^2+\frac{\xi_1}{\alpha_3}}\right) \tag{14}$$

$$b=\sqrt{\alpha_3}\left(\mu+v+\frac{\alpha_2}{\alpha_3}-1-\sqrt{\left(\frac{\alpha_2}{\alpha_3}-1\right)^2+\frac{\xi_1}{\alpha_3}}\right) \tag{15}$$

$$c=\alpha_1+2\mu \tag{16}$$

Solutions of the Schrodinger Equation for Exponential Screened Plus Yukawa Potential with Nikiforov-Uvarov Functional Analysis method

The Schrodinger equation in spherical coordinate with a confining potential is given by [42];

$$\frac{d^2\psi_{nlm}(r,\theta,\varphi)}{dr^2}+\frac{2\mu}{\hbar^2}\left[E_{nl}-V(r)-\frac{\hbar^2 l(l+1)}{2\mu r^2}\right]\psi_{nlm}(r,\theta,\varphi)=0 \tag{17}$$

where μ is the reduced mass, $\hbar = \frac{h}{2\pi}$ is the reduced Planck's constant, l is the orbital angular momentum quantum number, E_{nl} is the energy level of the system, V(r) is the interacting potential function and $\Psi_{nlm}(r,\theta,\varphi)$ is the a corresponding wavefunction.

The spherical symmetric of Eq. (4) allows us to write the wavefunction as,

$$\Psi(r,\theta,\varphi) = R_{nl}(r)Y_{lm}(\theta,\varphi) \tag{18}$$

where $Y_{lm}(\theta,\varphi)$ is the spherical harmonics and defined as,

$$Y_{lm}(\theta,\varphi) = \frac{1}{\sqrt{2\pi}}e^{im\varphi}\Phi_{lm}(\cos\theta) \tag{19}$$

Substituting the Exponential Screened plus Yukawa potential Eq. (1) into Eqs. (18) and (19) into Eq. (17), we obtain the radial Schrodinger wave equation as

$$\frac{dR_{nl}(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2}\left(E - D_0\left(1+e^{-2\alpha r}\right) - \frac{D_1 e^{-\alpha r}}{r}\right) - \frac{l(l+1)}{r^2}\right]R_{nl}(r) = 0 \tag{20}$$

Using Greene and Aldrich approximation scheme given as

$$\frac{1}{r^2} = \frac{4\alpha^2 e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2} \tag{21}$$

$$\frac{1}{r} = \frac{2\alpha e^{-\alpha r}}{1-e^{-2\alpha r}} \tag{22}$$

Substituting Eqs. (21) and (22) into (20),

$$\frac{d^2R_{nl}(r)}{dr^2} + \left[\frac{2\mu E}{\hbar^2} - \frac{2\mu}{\hbar^2}\left(D_0\left(1+e^{-2\alpha r}\right) + \frac{2D_1\alpha e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)}\right) - \frac{4\alpha^2 l(l+1)e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}\right]R(r) = 0 \tag{23}$$

Let

$$z = e^{-2\alpha r} \tag{24}$$

then,

$$\frac{d^2}{dr^2} = 4\alpha^2 z^2\frac{d^2}{dz^2} + 4\alpha^2 z\frac{d}{dz} \tag{25}$$

Substituting Eq. (25) into Eq. (23) we have

$$\frac{d^2R}{dz^2} + \frac{1}{z}\frac{(1-z)}{(1-z)}\frac{dR}{dz} + \frac{1}{z^2(1-z)^2}\begin{bmatrix} z^2\left(\varepsilon^2 + T_1 + T_2\right) + \\ z\left(-2\varepsilon^2 + T_1 - T_2 - l(l+1)\right) + \varepsilon^2 - T_1 \end{bmatrix} R(r) = 0 \tag{26}$$

where

$$\varepsilon^2 = \frac{2\mu E}{4\alpha^2\hbar^2}, T_1 = \frac{\mu D_0}{2\alpha^2\hbar^2}, T_2 = \frac{\mu D_1}{\alpha\hbar^2} \tag{27}$$

Comparing Eq. (26) with (3), we have the following

$$\begin{aligned} &\alpha_1 = \alpha_2 = \alpha_3 = 1 \\ &\xi_1 = -\varepsilon^2 - T_1 - T_2 \\ &\xi_2 = -2\varepsilon^2 + T_1 - T_2 - l(l+1), \\ &\xi_3 = -\varepsilon^2 + T_1 \end{aligned} \tag{28}$$

Using Eq. (9), μ is given by

$$\mu = \pm\sqrt{-\varepsilon^2 + T_1} \tag{29}$$

Also, using Eq. (10), v is given by

$$v = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4T_1 + 4l(l+1)} \tag{30}$$

RESULTS AND DISCUSSION

The energy spectrum for V(r) is obtained using Eq. (12) and is given by

$$\varepsilon^2 = \frac{\mu D_0}{2\alpha^2\hbar^2} - \left[\frac{\left(n + \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{2\mu D_0}{\alpha^2\hbar^2} + 4l(l+1)}\right)^2 + \frac{\mu D_0}{\alpha^2\hbar^2} + \frac{\mu D_1}{\alpha\hbar^2}}{2\left(n + \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{2\mu D_0}{\alpha^2\hbar^2} + 4l(l+1)}\right)}\right]^2 \tag{31a}$$

$$E_{nl} = D_0 - \frac{2\alpha^2\hbar^2}{\mu}\left[\frac{\left(n + \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{2\mu D_0}{\alpha^2\hbar^2} + 4l(l+1)}\right)^2 + \frac{\mu D_0}{\alpha^2\hbar^2} + \frac{\mu D_1}{\alpha\hbar^2}}{2\left(n + \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{2\mu D_0}{\alpha^2\hbar^2} + 4l(l+1)}\right)}\right]^2 \tag{31b}$$

Using Eq. (13), the wavefunction is given by

$$\psi(z) = N z^{\left(\sqrt{\frac{-2\mu E}{4\alpha^2\hbar^2} + \frac{\mu D_0}{2\alpha^2\hbar^2}}\right)} (1-z)^{\left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{2\mu D_0}{\alpha^2\hbar^2} + 4l(l+1)}\right)} {}_2F_1(a,b,c;z) \tag{32}$$

where

$$a = \pm\sqrt{\frac{-2\mu E}{4\alpha^2\hbar^2} + \frac{\mu D_0}{2\alpha^2\hbar^2}} + \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{\alpha^2\hbar^2 - 2\mu D_0}{\alpha^2\hbar^2}} + \sqrt{\frac{-2\mu E}{4\alpha^2\hbar^2} - \frac{\mu D_0}{2\alpha^2\hbar^2} - \frac{\mu D_1}{\alpha\hbar^2}} \tag{33}$$

$$b = \pm\sqrt{\frac{2\mu E}{4\alpha^2\hbar^2} + \frac{\mu D_0}{2\alpha^2\hbar^2}} + \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{\alpha^2\hbar^2 - 2\mu D_0}{\alpha^2\hbar^2}} - \sqrt{\frac{2\mu E}{4\alpha^2\hbar^2} - \frac{\mu D_0}{2\alpha^2\hbar^2} - \frac{\mu D_1}{\alpha\hbar^2}} \tag{34}$$

$$c = 1 \pm 2\sqrt{\frac{-2\mu E}{4\alpha^2\hbar^2} + \frac{\mu D_0}{2\alpha^2\hbar^2}} \tag{35}$$

In this work, we proposed an exponential screened plus Yukawa potential to determine an approximate solution of Schrodinger equation using NUFA. The results of our proposed potential reduce to some special cases which produce an excellent result as reported in the existing literature (Ikot *et al.*, 2021 and Ita*et al.*, 2017). The results for special cases of the proposed potential are applicable in molecular and high energy physics.

Exponential Screened Potential

Setting $D_0 = 0$ in Eq. (1), our proposed potential reduces to Yukawa potential of the form (Ikot *et al.*, 2021)

$$V(r) = D_0(1 + e^{-2\alpha r}). \tag{36}$$

The energy equation for the exponential Screened potential is obtained by setting $D_1 = 0$ in Eq. (31) as

$$E_{nl} = D_0 - \frac{2\alpha^2\hbar^2}{\mu}\left[\frac{\left(n + \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{2\mu D_0}{\alpha^2\hbar^2} + 4l(l+1)}\right)^2 + \frac{\mu D_0}{\alpha^2\hbar^2}}{2\left(n + \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{2\mu D_0}{\alpha^2\hbar^2} + 4l(l+1)}\right)}\right]^2. \tag{37}$$

And the corresponding wave function is expressed as

$$\psi(z)=Nz^{\left(\pm\sqrt{\frac{-2\mu E}{4\alpha^2\hbar^2}+\frac{\mu D_0}{2\alpha^2\hbar^2}}\right)}(1-z)^{\left(\frac{1}{2}+\frac{1}{2}\sqrt{\frac{2\mu D_0}{\alpha^2\hbar^2}+4l(l+1)}\right)}{}_2F_1(a,b,c;z). \tag{38}$$

Yukawa Potential

Setting $D_0 = 0$ in Eq. (1), our proposed potential reduces to Yukawa potential of the form (Ita *et al*., 2017)

$$V(r)=\frac{D_1 e^{-\alpha r}}{r} \quad . \tag{39}$$

Using Eq. (31) and setting $D_0 = 0$, the energy equation of the Yukawa potential is given by

$$E_{nl}=-\frac{2\alpha^2\hbar^2}{\mu}\left[\frac{\left(n+\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4l(l+1)}\right)^2+\frac{\mu D_1}{\alpha\hbar^2}}{2\left(n+\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4l(l+1)}\right)}\right]^2. \tag{40}$$

And the corresponding wavefunction is

$$\psi(z)=Nz^{\left(\sqrt{\frac{\mu D_0}{2\alpha^2\hbar^2}-\frac{2\mu E}{4\alpha^2\hbar^2}}\right)}(1-z)^{\left(\frac{1}{2}+\frac{1}{2}\sqrt{\frac{2\mu D_0}{\alpha^2\hbar^2}+4l(l+1)}\right)}{}_2F_1(a,b,c;z) \tag{41}$$

CONCLUSION

In this work, we determined an approximate solutions of Schrodinger equation with exponential screened plus Yukawa potential using Nikiforov-Uvarov functional analysis method with the help of Greene-Aldrich approximation to the centrifugal term. The total un-normalised wave function was obtained and expressed in terms of Jacobi-hypergeometric function. Our potential reduces exponential screened and Yukawa potential as special cases. The resulting energy eigen equation for the special case is in excellent agreement of work of an existing literature.

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