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APPROXIMATION OF COMMON FIXED POINTS OF A FINITE FAMILY OF LIPSCHITZ $\emptyset$ −DEMICONTRACTIVE MAPS USING A COMPOSITE IMPLICIT ITERATION PROCESS

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ABSTRACT: In this paper, the Composite Implicit Iteration Process recently introduced by Su and Li (2006) has been used to approximate the common fixed points of a finite family of Lipschitz $\emptyset$ −demicontractive maps in real Hilbert spaces. Our results complement the results of Su and Li (2006), Isiogugu(2005), Xu and Ori (2001) and generalize several others in the literature.

INTRODUCTION

Let H be a real Hilbert space and K a nonempty subset of H. A mapping $T: K \to K$ is said to be *demicontractive* (Hicks and Kubicek, 1979) if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$ and for all $x \in K, p \in F(T)$ and a constant $k > 0,$

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \qquad (1)$$

$k = 1 - 2\lambda > 0, \quad \lambda \ constant.$

T is called *strictly pseudocontractive* in the terminology of Browder and Petryshyn (1967) if there exists $x, y \in K, \ k \in [o, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2. \qquad (2)$$

Observe that a strictly pseudocontractive map with a nonempty fixed point set F(T) is demicontractive. An example of a demicontraction which is not strictly pseudocontractive was given by Hicks and Kubicek (1979). The class of demicontractive maps includes the important class of *quasi-nonexpansive* maps (i.e mappings such that $F(T) \neq \emptyset, and$

$\|Tx - p\| \leq \|x - p\|, \forall x \in D(T), p \in F(T)$). It was also shown by Hicks and Kubicek (1979) that the class of demicontractive maps includes the class of mappings studied by Kannan (1974) and Wong (1974) if the mappings have nonempty fixed point sets. Also if $F(T) \neq \emptyset$, then the class of demicontractive maps includes the class of mappings studied by Goebel *et al* (1973), since these mappings have been shown to be quasi-nonexpansive by Petryshyn and Williamson (1973). Furthermore, the class of demicontractive maps is equivalent to the class of mappings studied by Maruster (1977).

Again, T is L-Lipschitz if $\forall x, y \in D(T)$ there exists a constant $L > 0 \ni \|Tx - Ty\| \leq L\|x - y\|$.

In an arbitrary Banach Space E, a mapping $T: K \to K$ is said to be demicontractive (Hicks and Kubicek, 1979) if $F(T) \neq \emptyset$ and for all $x \in K$ and $p \in F(T) \exists$
$j(k - p) \in J(x - p)$ and a constant λ such that

$$\langle x - Tx, j(x-p)\rangle \geq \lambda \|x - Tx\|^2 \ , \tag{3}$$

where j is the single-valued duality mapping from E into E^* given by $j(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$ which holds in strictly convex Banach spaces, E* is the dual space of E and $\langle , \rangle$ denotes the generalized duality pairing.

T is said to be $\emptyset -$*demicontractive* (Isiogugu, 2005), if $F(T) \neq \emptyset$ and there exists a strictly increasing continuous function $\emptyset : [0,\infty) \to [0,\infty)$ with $\emptyset(0) = 0$ such that

$$\langle x - Tx, j(x-p) \geq \emptyset(\|x - Tx\|)\rangle. \tag{4}$$

The class of $\emptyset -$ demicontractive maps was first studied in arbitrary real Banach space by Isiogugu (2005). In Hilbert space, we shall call a mapping $T: K \to K$, $\emptyset -$*demicontractive* if $F(T) \neq \emptyset$ and there exists an increasing continuous function $\emptyset : [0,\infty) \to [0,\infty)$ with $\emptyset(0) = 0$ such that,

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \|x - Tx\|^2 - \emptyset(\|x - Tx\|), \forall\, x \in K \text{ and } p \in F(T). \tag{5}$$

Every demicontractive map is $\emptyset -$demicontractive with $\emptyset : [0,\infty) \to [0,\infty)$ given by $\emptyset(t) = \lambda t^2$.

The following example (Isiogugu, 2005) shows that the class of demicontractive maps is a proper sub-class of $\emptyset -$demicontractive maps.

Let $\Re$ denote the reals with the usual norm and let $K = (-\infty, 1)$. Define $T: K \to K$ by

$$Tx = \begin{cases} \frac{x}{1-x}, & -\infty < x \leq 0 \\ \frac{x}{x-1}, & 0 \leq x < 1. \end{cases}$$

Then $F(T) = \{0\}$ and if $p = 0$, then $|Tx - Tp| = \frac{|x|}{|1-x|} \leq |x - p|$.

Thus $|Tx - Tp|^2 = |x - p|^2 - 2\langle x - Tx, x - p\rangle + |x - Tx|^2 \leq |x - p|^2$.

So that

$$\langle x - Tx, x - p\rangle \geq \frac{1}{2}|x - Tx|^2 \geq \frac{|x-Tx|^2}{2+|x-Tx|}, \quad x \in (-\infty, 0] \text{ and } p \in F(T). \tag{6}$$

For every $x \in [0,1)$, we have $Tx = \frac{x}{x-1}$ and $x - Tx = \frac{x(2-x)}{1-x} = |x - Tx|$.

Thus, $2 + |x - Tx| = \frac{2-x^2}{1-x}$ and $\frac{|x-Tx|}{2+|x-Tx|} = \frac{x(2-x)}{2-x^2} \leq x = |x|$.

For $p = 0 \in F(T) = \{0\}$, we have:

$$\langle x - Tx, x - p\rangle = x(x - Tx) = |x||x - Tx| \geq \frac{|x-Tx|^2}{2+|x-Tx|}. \tag{7}$$

Equations (6) and (7) now imply that

$$\langle x-Tx,x-p\rangle \geq \frac{|x-Tx|^2}{2+|x-Tx|}\forall x \in (-\infty,1) \text{ and } \forall\, p \in F(T). \tag{8}$$

It follows that $\langle x-Tx,x-p\rangle \geq \emptyset(|x-Tx|)$,

where $\emptyset: [0,\infty) \to [0,\infty)$ is given by $\emptyset(t)=\frac{t^2}{2+t}$.

Clearly, $\emptyset$ is continuous, strictly increasing and $\emptyset(0)=0$. Hence, T is $\emptyset-$demicontractive. Given any $L>0$, if we choose $x \in \left(1-\frac{1}{L},1\right)$, then $|Tx-p|=\frac{x}{1-x}|x-p|>L|x-p|$.

Since every demicontractive mapping $T: D(T) \to H$ satisfies $\|Tx-p\| \leq L\|x-p\|$,

$\forall x \in D(T)\ p \in F(T)$ and some $L>0$ (see for example, Hicks and Kubicek, 1979; Chidume and Nnoli, 2002; Maruster, 1977), it follows that T is not demicontractive.

Isiogugu (2005) proved the convergence of the Mann(1953) iteration Scheme to the fixed points of $\emptyset-$demicontractive maps. Specifically she proved the following:

Theorem 1.1: (Isiogugu, 2005). Let E be a real Banach space and let K be a nonempty closed convex subset of E. Let $T: K \to K$ be a Lipschitz $\emptyset-$demicontractive map with Lipschitz constant L. Let $\{\alpha_n\}$ be a real sequence satisfying the conditions:

(i) $0<\alpha_n<1$ (ii) $\sum_{n=1}^{\infty}\alpha_n=\infty$ (iii) $\sum_{n=1}^{\infty}\alpha_n^2<\infty$.

Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by

$x_{n+1}=(1-\alpha_n)x_n+\alpha_n Tx_n, n \geq 1.$ Then

(i) $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p \in F(T)$. (ii) $\lim_{n\to\infty} inf\|x_n-Tx_n\|=0$.

(iii) $\{x_n\}$ converges strongly to a fixed point p of T if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges strongly to p.

Xu and Ori (2001) introduced an implicit iteration process and proved weak convergence theorem for approximation of common fixed points of a finite family of nonexpansive mappings (i.e. a subclass of strictly pseudocontractive mappings for which $\|Tx-Ty\| \leq \|x-y\|\ \forall x,y \in K$). Osilike (2004) extended the results of Xu and Ori (2001) from nonexpansive mappings to strictly pseudocontractive mappings. Motivated by the work of Xu and Ori, and Osilike, Su and Li (2006) introduced a new implicit iteration process and called it *Composite Implicit Iteration Process.* Using the new iteration process, they proved the results established by Osilike (2004). In compact form, the composite implicit iteration process of Su and Li (2006) is the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by

$$x_n=\alpha_n x_{n-1}+(1-\alpha_n)T_n y_n$$

$$y_n = \beta x_{n-1} + (1-\beta_n)T_n x_n, \quad n \geq 1 \tag{9}$$

where $\{\alpha_n\}, \{\beta_n\} \subseteq [0,1]$, $T_n = T_{n \bmod N}$.

Using (9), Su and Li (2006) proved the following:

Theorem 1.2 (Su and Li, 2006): Let E be a real Banach space and let K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N strictly pseudocontractive self-maps of K such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$, where $F(T_i) = \{x \in K: T_i x = x\}$ and let $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty \subset [0,1]$ be two real sequences satisfying the conditions:

(i) $\sum_{n=1}^\infty (1-\alpha_n) = +\infty$ (ii) $\sum_{n=1}^\infty (1-\alpha_n)^2 < +\infty$

(iii) $\sum_{n=1}^\infty (1-\beta_n) < +\infty$ (iv) $(1-\alpha_n)(1-\beta_n)L^2 < 1, \forall n \geq 1$,

where $L \geq 1$ is common Lipschitz constant of $\{T_i\}_{i=1}^N$.

For $x_1 \in K$, let $\{x_n\}_{n=1}^\infty$ be defined by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n y_n$$
$$y_n = \beta_n x_{n-1} + (1-\beta_n)T_n x_n, \tag{10}$$

where $T_n = T_{n \bmod N}$. Then

(i) $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$. (ii) $\lim_{n\to\infty} \inf \|x_n - T_n x_n\| = 0$.

Observe that if $T: K \to K$ is $L-$Lipschitzian $\emptyset-$demicontractive map, then for every fixed $u \in K$ and $t \in \left(\frac{L}{1+L}, 1\right)$ the operator $S_{t,s,n}: K \to K$ defined for all $x \in K$ by $S_{t,s,n}\ x = tu + (1-t)T[Su + (1-S)Tx]$ satisfies

$$\|S_{t,s,n}\ x - S_{t,s,n}\ y\| \leq (1-t)(1-s)L^2\|x-y\|, \forall\, x, y \in K.$$

Since $(1-t)(1-s)L^2 \in (0,1)$, it follows that $S_{t,s,n}$ is a contraction and hence has a unique fixed point $x_{t,s,n}$ in K. this implies that there exists a unique $x_{t,s,n} \in K$ such that $x_{t,s,n} = tu + (1-t)T[Su + (1-S)Tx]$. Thus, the iteration process (9) is defined in K for the family $\{T_i\}_{i=1}^N$ of N L_i-Lipschitzian $\emptyset-$demicontractive mappings of nonempty convex subset K of a Hilbert space provided that $\{\alpha_n\}, \{\beta_n\} \subseteq (\eta, 1)$ for all $n \geq 1$,

where $\eta = \frac{L}{1+L}$ and $L = \max_{1 \leq i \leq N}\{L_i\}$.

The purpose of this paper is to first study the class of $\emptyset-$demicontractive maps in real Hilbert spaces. Thereafter, we investigate the problem of approximating common fixed points of $\emptyset-$demicontractive maps in real Hilbert spaces by the iteration sequence (9) recently introduced by Su and Li (2006). The results of this paper complement, generalize or extend the results of Hicks and Kubicek (1979), Isiogugu (2005), Osilike (2004), Xu and Ori (2001), Su and Li (2006) and several others in the literature.

In the sequel , we need the following:

Lemma 1.1 (Osilike and Aniagbosor, 2002): Let $\{a_n\},\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality,

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

MAIN RESULTS

Let K be a subset of a real Hilbert space H . We call the mapping $T: K \to K$ $\phi - demicontractive$ map if $F(T) \neq \phi$ and there exists a strictly increasing continuous function $\emptyset: [0,\infty) \to [0,\infty)$ with $\phi(0)=0$ such that

$$\|Tx-Tp\|^2 = \|x-p\|^2 + \|x-Tx\|^2 - \phi(\|x-Tx\|),\ \forall x \in K \text{ and } p \in F(T). \tag{11}$$

. Since in a Hilbert space, j is an identity map, using equation (4) we have:

$$\|Tx-Tp\|^2 \leq \|x-p-[(I-T)x-(I-T)p]\|^2$$

$$= \|x-p\|^2 - 2\langle x-Tx,(x-p)\rangle + \|x-Tx\|^2$$

$$\leq \|x-p\|^2 + \|x-Tx\|^2 - \phi(\|x-Tx\|),$$

proving (11).

Theorem 2.1: Let H be a real Hilbert space and Let K be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^{N}$ be N L_i-Lipschitz $\phi -$demicontractive self maps of K such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$ where $F(T_i) = \{x \in K : T_i x = x\}$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \subset [\eta,1]$ be two real sequences satisfying the conditions:

(i) $\sum_{n=1}^{\infty}(1-\alpha_n) = +\infty$ (ii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2 < +\infty$ (iii) $\sum_{n=1}^{\infty}(1-\beta_n) < +\infty$.

(iv) $(1-\alpha_n)(1-\beta_n)L^2 < 1,\ 0 < \beta \leq \beta_n \leq \alpha \leq 1,$

where $\eta = \dfrac{L}{1+L}$ and $L = \max\{L_i\}$, L_i Lipschitz constant of T_i, $i = 1,\ldots,N$.

For $x_1 \in K$, let $\{x_n\}_{n=1}^{\infty}$ be generated by

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)T_n y_n \\ y_n &= \beta_n x_{n-1} + (1-\beta_n)T_n x_n, \quad n \geq 1 \end{aligned} \tag{12}$$

where $T_n = T_{n \bmod N}$, then

(a) $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$. (b) $\lim_{n\to\infty} \inf\|x_n - T_n x_n\| = 0.$

(c) $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point p of the mapping $\{T_i\}_{i=1}^{N}$ if there is a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which converges strongly to p.

Proof: Indeed, since T_i is Lipschitzian for each i and every Hilbert space H is uniformly convex, by the theorem of Goebel and Kirk (1973), $F(T_i) \neq \emptyset$ *for each i.* Let $p \in F(T)$. We use the well known result of Reinermann (1969), Osilike and Igbokwe, 2000).

$$\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \tag{13}$$

which holds for all $x,\ y \in H,\ t \in [0,1]$.

Using equation(12) and equation(13), we obtain:

$$\begin{aligned}
\|x_n - p\|^2 &= \|\alpha_n(x_{n-1}-p)+(1-\alpha_n)(T_n y_n - p)\|^2 \\
&= \alpha_n\|x_{n-1}-p\|^2 + (1-\alpha_n)\|T_n y_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_{n-1}-T_n y_n\|^2 \\
&\leq \alpha_n\|x_{n-1}-p\|^2 + L^2(1-\alpha_n)\|y_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_{n-1}-T_n y_n\|^2 \\
&\leq \alpha_n\|x_{n-1}-p\|^2 + L^2(1-\alpha_n)\|y_n - p\|^2 + \alpha_n(1-\alpha_n)\|x_{n-1}-T_n y_n\|^2
\end{aligned}$$

$$= \alpha_n\|x_{n-1}-p\|^2 + L^2(1-\alpha_n)\{\ \|\beta_n(x_{n-1}-p)+(1-\beta_n)(T_n x_n - p)\|^2\} + \alpha_n(1-\alpha_n)\|x_{n-1}-T_n y_n\|^2$$

$$\leq \alpha_n\|x_{n-1}-p\|^2 + L^2(1-\alpha_n)\{\ \beta_n\|x_{n-1}-p\|^2$$
$$+(1-\beta_n)\|T_n x_n - p\|^2 - \beta_n(1-\beta_n)\|x_{n-1}-T_n x_n\|^2\ \}$$
$$+\alpha_n(1-\alpha_n)\|x_{n-1}-T_n y_n\|^2$$

$$\begin{aligned}
&= \alpha_n\|x_{n-1}-p\|^2 + L^2\beta_n(1-\alpha_n)\|x_{n-1}-p\|^2 + L^2(1-\alpha_n)(1-\beta_n)\|T_n x_n - p\|^2 \\
&\quad - L^2\beta_n(1-\alpha_n)_n(1-\beta_n)\|x_{n-1}-T_n x_n\|^2 + \alpha_n(1-\alpha_n)\|x_{n-1}-T_n y_n\|^2 \\
&\leq \alpha_n\|x_{n-1}-p\|^2 + L^2\beta_n(1-\alpha_n)\|x_{n-1}-p\|^2 + L^2(1-\alpha_n)(1-\beta_n)\|T_n x_n - p\|^2 \\
&\quad - L^2\beta_n(1-\alpha_n)_n(1-\beta_n)\|x_{n-1}-T_n x_n\|^2 + (1-\alpha_n)\|x_{n-1}-T_n y_n\|^2.
\end{aligned} \tag{14}$$

Observe that since T_i is L_i-Lipschitzian,

$$\begin{aligned}
\|x_{n-1}-T_n y_n\| &\leq \|x_{n-1}-p\| + \|T_n y_n - p\| \\
&\leq \|x_{n-1}-p\| + L\|y_n - p\| \\
&= \|x_{n-1}-p\| + L\|\beta_n(x_{n-1}-p)+(1-\beta_n)(T_n x_n - p)\| \\
&\leq \|x_{n-1}-p\| + L\beta_n\|x_{n-1}-p\| + L^2(1-\beta_n)\|x_n - p\|.
\end{aligned}$$

Hence,

$$\|x_{n-1} - T_n y_n\|^2 \le \{(L\beta_n + 1)\|x_{n-1} - p\| + L^2(1-\beta_n)\|x_n - p\|\}^2$$

$$= (L\beta_n + 1)^2\|x_{n-1} - p\|^2 + 2L^2(1-\beta_n)(L\beta_n + 1)\|x_{n-1} - p\|\|x_n - p\|$$

$$+ L^4(1-\beta_n)^2\|x_n - p\|^2.$$

Since $2\|x_{n-1} - p\|\|x_n - p\| \le \|x_{n-1} - p\|^2 + \|x_n - p\|^2$, we have

$$\|x_{n-1} - T_n y_n\|^2 \le (L\beta_n + 1)^2\|x_{n-1} - p\|^2 + L^2(1-\beta_n)(L\beta_n + 1)\{\|x_{n-1} - p\|^2 + \|x_n - p\|^2\}$$

$$+ L^4(1-\beta_n)^2\|x_n - p\|^2$$

$$= \left[(L\beta_n + 1)^2 + L^2(1-\beta_n)(L\beta_n + 1)\right]\|x_{n-1} - p\|^2$$

$$+ \left[L^2(1-\beta_n)(L\beta_n + 1) + L^4(1-\beta_n)^2\right]\|x_n - p\|^2. \tag{15}$$

Substitute equation (15) into equation(14) to obtain:

$$\|x_n - p\|^2 \le \alpha_n\|x_{n-1} - p\|^2 + L^2\beta_n(1-\alpha_n)\|x_{n-1} - p\|^2 + L^2(1-\alpha_n)(1-\beta_n)\|T_n x_n - p\|^2$$

$$- L^2\beta_n(1-\alpha_n)(1-\beta_n)\|x_{n-1} - T_n x_n\|^2$$

$$+ (1-\alpha_n)\{ \left[(L\beta_n + 1)^2 + L^2(1-\beta_n)(L\beta_n + 1)\right]\|x_{n-1} - p\|^2$$

$$+ \left[L^2(1-\beta_n)(L\beta_n + 1) + L^4(1-\beta_n)^2\right]\|x_n - p\|^2 \}$$

$$= \alpha_n\|x_{n-1} - p\|^2 + L^2\beta_n(1-\alpha_n)\|x_{n-1} - p\|^2 + L^2(1-\alpha_n)(1-\beta_n)\|T_n x_n - p\|^2$$

$$- L^2\beta_n(1-\alpha_n)(1-\beta_n)\|x_{n-1} - T_n x_n\|^2$$

$$+ (1-\alpha_n)\left[(L\beta_n + 1)^2 + L^2(1-\beta_n)(L\beta_n + 1)\right]\|x_{n-1} - p\|^2$$

$$+ (1-\alpha_n)\left[L^2(1-\beta_n)(L\beta_n + 1) + L^4(1-\beta_n)^2\right]\|x_n - p\|^2$$

$$= [\alpha_n + L^2\beta_n(1-\alpha_n) + (1-\alpha_n)\left[(L\beta_n + 1)^2 + L^2(1-\beta_n)(L\beta_n + 1)\right]]\|x_{n-1} - p\|^2$$

$$+ L^2(1-\alpha_n)(1-\beta_n)\|T_n x_n - p\|^2$$

$$-L^2\beta_n(1-\alpha_n)(1-\beta_n)\|x_{n-1}-T_nx_n\|^2$$

$$+(1-\alpha_n)\left[L^2(1-\beta_n)(L\beta_n+1)+L^4(1-\beta_n)^2\right]\|x_n-p\|^2. \tag{16}$$

Using equation(11), we have:

$$\|T_nx_n-Tp\|^2 \leq \|x_n-p\|^2+\|x_n-T_nx_n\|^2-\phi(\|x_n-T_nx_n\|)$$

$$\leq \|x_n-p\|^2+(L+1)^2\|x_n-p\|^2-\phi(\|x_n-T_nx_n\|)$$

$$= \left[(L+1)^2+1\right]\|x_n-p\|^2-\phi(\|x_n-T_nx_n\|) \tag{17}$$

Substitute equation(17) into equation(16) to obtain:

$$\|x_n-p\|^2 \leq [\alpha_n+L^2\beta_n(1-\alpha_n)+(1-\alpha_n)[(L\beta_n+1)^2+L^2(1-\beta_n)(L\beta_n+1)]]\|x_{n-1}-p\|^2$$

$$+L^2(1-\alpha_n)(1-\beta_n)\{[(L+1)^2+1]\|x_n-p\|^2-\phi(\|x_n-T_nx_n\|)\}$$

$$-L^2\beta_n(1-\alpha_n)(1-\beta_n)\|x_{n-1}-T_nx_n\|^2$$

$$+(1-\alpha_n)\left[L^2(1-\beta_n)(L\beta_n+1)+L^4(1-\beta_n)^2\right]\|x_n-p\|^2$$

$$= [\alpha_n+L^2\beta_n(1-\alpha_n)+(1-\alpha_n)[(L\beta_n+1)^2+L^2(1-\beta_n)(L\beta_n+1)]]\|x_{n-1}-p\|^2$$

$$-L^2(1-\alpha_n)(1-\beta_n)\phi(\|x_n-T_nx_n\|) \quad -L^2\beta_n(1-\alpha_n)(1-\beta_n)\|x_{n-1}-T_nx_n\|^2$$

$$+(1-\alpha_n)\left[L^2(1-\beta_n)(L\beta_n+1)+L^4(1-\beta_n)^2+L^2(1-\beta_n)[(L+1)^2+1]\right]\|x_n-p\|^2.$$

$$\left[1-(1-\alpha_n)\left[L^2(1-\beta_n)(L\beta_n+1)+L^4(1-\beta_n)^2+L^2(1-\beta_n)[(L+1)^2+1]\right]\right]\|x_n-p\|^2 \leq$$

$$[\alpha_n+L^2\beta_n(1-\alpha_n)+(1-\alpha_n)[(L\beta_n+1)^2+L^2(1-\beta_n)(L\beta_n+1)]]\|x_{n-1}-p\|^2$$

$$-L^2(1-\alpha_n)(1-\beta_n)\phi(\|x_n-T_nx_n\|) \quad -L^2\beta_n(1-\alpha_n)(1-\beta_n)\|x_{n-1}-T_nx_n\|^2.$$

$$\left[1-L^2(1-\beta_n)(L\beta_n+1)-L^4(1-\beta_n)^2-L^2(1-\beta_n)[(L+1)^2+1]\right]\|x_n-p\|^2 \leq$$

$$[\alpha_n+L^2\beta_n+(L\beta_n+1)^2+L^2(1-\beta_n)(L\beta_n+1)]\|x_{n-1}-p\|^2$$

$$-L^2(1-\alpha_n)(1-\beta_n)\phi(\|x_n-T_nx_n\|) \quad -L^2\beta_n(1-\alpha_n)(1-\beta_n)\|x_{n-1}-T_nx_n\|^2.$$

$$\left[(1-\beta_n)-(1-\beta_n)^2\left[L^2(L\beta_n+1)+L^4(1-\beta_n)+L^2[(L+1)^2+1]\right]\right]\|x_n-p\|^2 \leq$$

$$(1-\beta_n)[\alpha_n + L^2\beta_n + (L\beta_n+1)^2 + L^2(1-\beta_n)(L\beta_n+1)]\|x_{n-1}-p\|^2$$

$$-L^2(1-\alpha_n)(1-\beta_n)^2\phi(\|x_n - T_n x_n\|) \ -L^2\beta_n(1-\alpha_n)(1-\beta_n)^2\|x_{n-1}-T_n x_n\|^2.$$

$$\left[1-(1-\beta_n)^2\left[L^2(L\beta_n+1)+L^4(1-\beta_n)+L^2((L+1)^2+1)\right]-\beta_n\right]\|x_n-p\|^2 \le$$

$$(1-\beta_n)[\alpha_n + L^2\beta_n + (L\beta_n+1)^2 + L^2(1-\beta_n)(L\beta_n+1)]\|x_{n-1}-p\|^2$$

$$-L^2(1-\alpha_n)(1-\beta_n)^2\phi(\|x_n - T_n x_n\|) \ -L^2\beta_n(1-\alpha_n)(1-\beta_n)^2\|x_{n-1}-T_n x_n\|^2.$$

$$\{\ 1-(1-\beta_n)\left[L^2(L\beta_n+1)+L^4(1-\beta_n)+L^2((L+1)^2+1)\right]$$

$$+(1-\beta_n)\left[L^2(L\beta_n+1)+L^4(1-\beta_n)+L^2((L+1)^2+1)\right]\beta_n-\beta_n\ \}\|x_n-p\|^2 \le$$

$$(1-\beta_n)[\alpha_n + L^2\beta_n + (L\beta_n+1)^2 + L^2(1-\beta_n)(L\beta_n+1)]\|x_{n-1}-p\|^2$$

$$-L^2(1-\alpha_n)(1-\beta_n)^2\phi(\|x_n - T_n x_n\|) \ -L^2\beta_n(1-\alpha_n)(1-\beta_n)^2\|x_{n-1}-T_n x_n\|^2. \qquad (18)$$

Observed that $L^2(L\beta_n+1)+L^4(1-\beta_n)+L^2((L+1)^2+1)>1$. Using the fact that $(1-\beta_n)\le 1$, then, $(1-\beta_n)\times\left\{(1-\beta_n)\left[L^2(L\beta_n+1)+L^4(1-\beta_n)+L^2((L+1)^2+1)\right]\beta_n-\beta_n\right\}$ $\le$

$$(1-\beta_n)\left[L^2(L\beta_n+1)+L^4(1-\beta_n)+L^2((L+1)^2+1)\right]\beta_n-\beta_n$$

So we obtain:

$$\{\ 1-(1-\beta_n)\left[L^2(L\beta_n+1)+L^4(1-\beta_n)+L^2((L+1)^2+1)\right]$$

$$+(1-\beta_n)^2\beta_n\left[L^2(L\beta_n+1)+L^4(1-\beta_n)+L^2((L+1)^2+1)\right]-(1-\beta_n)\beta_n\ \}\|x_n-p\|^2 \le$$

$$(1-\beta_n)[\alpha_n + L^2\beta_n + (L\beta_n+1)^2 + L^2(1-\beta_n)(L\beta_n+1)]\|x_{n-1}-p\|^2$$

$$-L^2(1-\alpha_n)(1-\beta_n)^2\phi(\|x_n - T_n x_n\|) \ -L^2\beta_n(1-\alpha_n)(1-\beta_n)^2\|x_{n-1}-T_n x_n\|^2.$$

$$\left[1-(1-\beta_n)\lambda+(1-\beta_n)^2\beta_n\lambda-(1-\beta_n)\beta_n\right]\|x_n-p\|^2 \le$$

$$(1-\beta_n)[\alpha_n + L^2\beta_n + (L\beta_n+1)^2 + L^2(1-\beta_n)(L\beta_n+1)]\|x_{n-1}-p\|^2$$

$$-L^2(1-\alpha_n)(1-\beta_n)^2\phi(\|x_n - T_n x_n\|) \ -L^2\beta_n(1-\alpha_n)(1-\beta_n)^2\|x_{n-1}-T_n x_n\|^2,$$

where $\lambda_n = L^2(L\beta_n+1)+L^4(1-\beta_n)+L^2((L+1)^2+1)$.

So that

$$\|x_n - p\|^2 \le \frac{(1-\beta_n)[\alpha_n + L^2\beta_n + (L\beta_n+1)^2 + L^2(1-\beta_n)(L\beta_n+1)]}{1-(1-\beta_n)\lambda + (1-\beta_n)^2\beta_n\lambda - (1-\beta_n)\beta_n}\|x_{n-1} - p\|^2$$

$$-\frac{L^2(1-\alpha_n)(1-\beta_n)^2}{1-(1-\beta_n)\lambda + (1-\beta_n)^2\beta_n\lambda - (1-\beta_n)\beta_n}\phi(\|x_n - T_n x_n\|)$$

$$-\frac{L^2\beta_n(1-\alpha_n)(1-\beta_n)^2}{1-(1-\beta_n)\lambda + (1-\beta_n)^2\beta_n\lambda - (1-\beta_n)\beta_n}\|x_{n-1} - T_n x_n\|^2.$$

$$\le \frac{(1-\beta_n) + (1-\beta_n)\{L^2\beta_n + (L\beta_n+1)^2 + L^2(1-\beta_n)(L\beta_n+1)\}}{1-(1-\beta_n)\lambda + (1-\beta_n)^2\beta_n\lambda - (1-\beta_n)\beta_n}\|x_{n-1} - p\|^2$$

$$-\frac{L^2(1-\alpha_n)(1-\beta_n)^2}{1-(1-\beta_n)\lambda + (1-\beta_n)^2\lambda - (1-\beta_n)\beta_n}\phi(\|x_n - T_n x_n\|)$$

$$-\frac{L^2\beta_n(1-\alpha_n)(1-\beta_n)^2}{1-(1-\beta_n)\lambda + (1-\beta_n)^2\beta_n\lambda - (1-\beta_n)\beta_n}\|x_{n-1} - T_n x_n\|^2.$$

$$= \left\{1 + \frac{\begin{matrix}(1-\beta_n) + (1-\beta_n)\{L^2\beta_n + (L\beta_n+1)^2 + L^2(1-\beta_n)(L\beta_n+1)\} \\ -[1-(1-\beta_n)\lambda + (1-\beta_n)^2\beta_n\lambda - (1-\beta_n)\beta_n]\end{matrix}}{1-(1-\beta_n)\lambda + (1-\beta_n)^2\beta_n\lambda - (1-\beta_n)\beta_n}\right\}\|x_{n-1} - p\|^2$$

$$-\frac{L^2(1-\alpha_n)(1-\beta_n)^2}{1-(1-\beta_n)\lambda + (1-\beta_n)^2\beta_n\lambda - (1-\beta_n)\beta_n}\phi(\|x_n - T_n x_n\|)$$

$$-\frac{L^2\beta_n(1-\alpha_n)(1-\beta_n)^2}{1-(1-\beta_n)\lambda + (1-\beta_n)^2\beta_n\lambda - (1-\beta_n)\beta_n}\|x_{n-1} - T_n x_n\|^2.$$

$$\le \left\{1 + \frac{\begin{matrix}(1-\beta_n) + (1-\beta_n)\{L^2\beta_n + (L\beta_n+1)^2 + L^2(1-\beta_n)(L\beta_n+1)\} \\ -1 + (1-\beta_n)\lambda - (1-\beta_n)^2\beta_n\lambda + \beta_n\end{matrix}}{1-(1-\beta_n)\lambda + (1-\beta_n)^2\beta_n\lambda - (1-\beta_n)\beta_n}\right\}\|x_{n-1} - p\|^2$$

$$-\frac{L^2(1-\alpha_n)(1-\beta_n)^2}{1-(1-\beta_n)\lambda + (1-\beta_n)^2\beta_n\lambda - (1-\beta_n)\beta_n}\phi(\|x_n - T_n x_n\|)$$

$$-\frac{L^2\beta_n(1-\alpha_n)(1-\beta_n)^2}{1-(1-\beta_n)\lambda+(1-\beta_n)^2\beta_n\lambda-(1-\beta_n)\beta_n}\|x_{n-1}-T_nx_n\|^2.$$

$$=\left\{1+\frac{\begin{array}{c}(1-\beta_n)\{L^2\beta_n+(L\beta_n+1)^2+L^2(1-\beta_n)(L\beta_n+1)\}\\+(1-\beta_n)\lambda-(1-\beta_n)^2\beta_n\lambda\end{array}}{1-(1-\beta_n)[\lambda(1-(1-\beta_n)\beta_n)+\beta_n]}\right\}\|x_{n-1}-p\|^2$$

$$-\frac{L^2(1-\alpha_n)(1-\beta_n)^2}{1-(1-\beta_n)[\lambda(1-(1-\beta_n)\beta_n)+\beta_n]}\phi(\|x_n-T_nx_n\|)$$

$$-\frac{L^2\beta_n(1-\alpha_n)(1-\beta_n)^2}{1-(1-\beta_n)[\lambda(1-(1-\beta_n)\beta_n)+\beta_n]}\|x_{n-1}-T_nx_n\|^2. \qquad (19)$$

Since $\lim_{n\to\infty}(1-\beta_n)=0,$ then there exists a natural number N_1, such that $\forall\ n>N_1$

$$1-(1-\beta_n)[\lambda(1-(1-\beta_n)\beta_n)+\beta_n]\geq\frac{1}{2}.$$

Therefore, it follows from equation(19) that:

$$\|x_n-p\|^2 \leq [1+2[(1-\beta_n)\{L^2\beta_n+(L\beta_n+1)^2+L^2(1-\beta_n)(L\beta_n+1)\}$$
$$+(1-\beta_n)\lambda-(1-\beta_n)^2\beta_n\lambda]\|x_{n-1}-p\|^2-L^2(1-\alpha_n)(1-\beta_n)^2\,\phi(\|x_n-T_nx_n\|)$$
$$-L^2\beta_n(1-\alpha_n)(1-\beta_n)^2\|x_{n-1}-T_nx_n\|^2$$
$$=[1+\delta_n]\|x_{n-1}-p\|^2-L^2(1-\alpha_n)(1-\beta_n)^2\phi(\|x_n-T_nx_n\|)$$
$$-L^2\beta_n(1-\alpha_n)(1-\beta_n)^2\|x_{n-1}-T_nx_n\|^2, \qquad (20)$$

where,

$$\delta_n=2[(1-\beta_n)\{L^2\beta_n+(L\beta_n+1)^2+L^2(1-\beta_n)(L\beta_n+1)\}+(1-\beta_n)\lambda-(1-\beta_n)^2\beta_n\lambda]$$

Since $\sum_{n=1}^{\infty}\delta_n<\infty$, it follows from equation(20) and Lemma 1.1 that $\underset{n\to\infty}{Lim}\|x_{n-1}-p\|$ exists. Hence $\underset{n\to\infty}{Lim}\|x_n-p\|$ exists. Thus completing the proof of (a).

Therefore, there exists $M>0$ such that $\|x_n-p\|\leq M,\ \forall n\geq 1.$

Since $\|x_n-p\|\leq M$, $\forall\ n\geq 1$ and some $M>0$, we obtain from equation(20) that

$$L^2(1-\alpha_n)(1-\beta_n)^2\emptyset(\|x_n-T_xx_n\|)\leq[1+\delta_n]\|x_{n-1}-p\|^2-\|x_n-p\|^2$$
$$\leq \|x_{n-1}-p\|^2-\|x_n-p\|^2+M^2\delta_n$$

$$L^2(1-\beta)^2 \sum_{j=N+1}(1-\alpha_j)\emptyset(\|x_j - T_j x_j\|) \leq \|x_N - p\|^2 + M^2 \sum_{j=N+1} \delta_j$$

$$\sum_{n=1}^{\infty}(1-\alpha_n)\emptyset(\|x_n - T_n x_n\|) < \infty.$$

Since $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, then we have $\underset{n\to\infty}{Lim} \inf \phi \|x_n - T_n x_n\| = 0.$

Since ϕ is an increasing and continuous function, then $\underset{n\to\infty}{Lim} \inf\|x_n - T_n x_n\| = 0.$
Thus completing the proof of (b).

Since $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ which converges strongly to p and $\underset{n\to\infty}{Lim}\|x_n - p\|$ exists by Lemma 1.1, $\underset{n\to\infty}{Lim}\|x_n - p\| = 0$.
Thus completing the proof.

Since every demicontractive map T is $\emptyset$ −demicontractive (Isiogugo, 2005), we have the following:

Corllary 2.1: Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^{N}$ be N L_i − Lipschitzian demicontractive self map of K such that

$$F = \cap_{i=1}^{N} F(T_i) \neq \emptyset \text{ where } F(T_i) = \{x \in K : T_i x = x\}.$$

Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset (\eta, 1)$ be two real sequences satisfying the conditions:

(i) $\sum_{n=1}^{\infty}(1-\alpha_n) = +\infty$ (ii) $\sum_{i=1}^{\infty}(1-\alpha_n)^2 < \infty$ (iii) $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$

(iv) $(1-\alpha_n)(1-\beta_n)L^2 < 1, \ \ 0 < \beta \leq \beta_n \leq \alpha \leq 1,$

where $\eta = \frac{L}{1+L}$ and $L = max\{L_i\}$, L_i −Lipschitzian constant of $T_i, i = 1,2, \ldots, N.$

For $x_1 \in K$, let $\{x_1\}$ be generated by (12), then

(a) $\lim_{n\to\infty}\|x_n - p\|$ exists (b) $\lim_{n\to\infty} inf\|x_n - T_n x_n\| = 0$

(c) $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point p of the mapping $\{T_i\}_{i=1}^{N}$ if there exists a subsequence $\{x_{nj}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which converges strongly to p.

Remark 1: If we set $\beta_n = 1$, then our iteration scheme (equation(12)) takes the non-implicit form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_i x_{n-1}, n \geq 1 . \qquad (21)$$

In the case of $N = 1, (21)$ becomes the Mann iteration process given by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T x_{n-1}. \tag{22}$$

The conclusions of Theorem 2.1 and Corollary 2.1 are still valid. Hence we state the following theorem without proof.

Theorem 2.2: Let K be a nonempty closed convex subset of a real Hilbert space H and T be an $L -$Lipschitzian $\emptyset -$demicontractive self-map of K such that $F = F(T) = \{x \in K : Tx = x\} \neq \emptyset$. Let $\{\alpha_n\} \subseteq (0,1)$ be a real sequence satisfying the conditions

(i) $\sum_{n=1}^{\infty}(1 - \alpha_n) = +\infty$ (ii) $\sum_{n=1}^{\infty}(1 - \alpha_n)^2 < +\infty$,

$$0 < \alpha \leq \alpha_n < \beta < 1.$$

For arbitrary $x_1 \in K$, let $\{x_n\}$ be the Mann iteration process generated by equation (2.2). Then,

(a) $\lim_{n\to\infty} \|x_n - p\|$ exists. (b) $\lim_{n\to\infty} inf \|x_n - T_n x_n\| = 0$ and

(c) $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point p of the mapping T if there exists a subsequence $\{x_{nj}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which converges strongly to p.

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