QUANTUM MECHANICS OF A RADIOACTIVE SUBSTANCE AS A NON CONSERVATIVE SYSTEM

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ITUEN, E. E, IKOT, A. N., ANTIA, A. D. AND AWOGA, O. A.
Department of Physics, University of Uyo,
P. M. B. 1017, Uyo – Nigeria.

ABSTRACT: The quantum mechanics of a radioactive substance, as a non conservative system, with a radioactive mass has been evaluated to obtain the analytical solutions. We also investigated the case of under-damped, over-damped and critically damped oscillations for this model. The results are respectively as obtained in equations 24, 29 and 32, which show that the under-damped oscillation clearly tends to agree with Heisenberg uncertainty principle

INTRODUCTION

A radioactive substance is a type of Caldirola-Kanai model (Caldirola, 1941 and Kanai, 1948) with mass $m(t) = m_0 e^{-\lambda t}$ that has a damping coefficient, λ, which is actually the usual decaying constant. Thus it is another case of non conservative system with Hamiltonian quadratic in co-ordinate and momentum operators (Ikot *et al*, 2008 and Ikot *et al*, 2010). This system also poses the problem of quantum oscillator with time-varying frequency which had been solved by Um *et al* (1987).

Recently Antia (2010) developed two similar models namely harmonic oscillator with pulsating mass and Caldirola-Kanai oscillator with modified damping factor. From the results of the latter case, we have the following:

Harmonic Oscillator with Pulsating Mass of $m(t) = m_0 cosh^2 \frac{\gamma t}{2}$ gives Hamiltonian and Lagrangian respectively as

$$\hat{H} = Cosh^{-2}\left(\frac{\gamma t}{2}\right)\frac{\hat{p}^2}{2m_0} + \left(\frac{1}{2}m_0 Cosh^2\left(\frac{\gamma t}{2}\right)\right)\omega^2(t)q^2$$

and $$L = \frac{1}{2}m_0 Cosh^2\left(\frac{\gamma t}{2}\right)\left[\dot{q}^2 - \omega^2(t)q^2\right]$$

Its equation of motion is $$\ddot{q}(t) + \left[\gamma \tanh\left(\frac{\gamma t}{2}\right)\right]\dot{q}(t) + \omega^2(t)q = 0$$

with solution: $$q(t) = e^{-\left(\frac{\gamma t}{2}\tanh\frac{\gamma t}{2}\right)}\left[Ae^{i\Omega t} + Be^{-i\Omega t}\right],$$

where $\Omega(t) = \sqrt{\omega^2(t) - \frac{\gamma^2}{4}\tanh^2\frac{\gamma t}{2}}$

Caldirola-Kanai oscillator with modified damping factor $m(t) = m_0 e^{sin\gamma t + \gamma t}$ gives Hamiltonian, Lagrangian and its equation of motion respectively as

$$\hat{H} = \frac{e^{-(Sin\,\gamma t+\gamma t)}}{2m_0}\hat{p}^2 + \frac{1}{2}e^{(Sin\,\gamma t+\gamma t)}m_0\omega^2(t)\hat{q}^2$$

$$L(q,\dot{q},t) = e^{(Sin\,\gamma t+\gamma t)}\frac{m_0}{2}\left(\dot{q}^2 - \omega^2(t)q^2\right)$$

and

$$\ddot{q}(t) + (\gamma\cos\gamma t + \gamma)\dot{q}(t) + \omega^2(t)q = 0$$

With $q(t) = e^{\frac{-(\gamma Cos\,\gamma t+\gamma t)}{2}}\left[Ae^{i\Omega t} + Be^{-i\Omega t}\right],$

Where $\Omega(t) = \sqrt{\omega^2(t) - \left(\frac{\gamma Cos\,\gamma t + \gamma}{2}\right)^2}$

RADIOACTIVE SUBSTANCE MODEL;

$$m(t) = m_0e^{-\lambda t}$$

The time-dependent Hamiltonian with time varying frequency for this model can, similarly, be obtained as:

$$\hat{H} = \frac{\hat{p}^2}{2m_0}\frac{}{e^{-\lambda t}} + \left(\frac{1}{2}m_0e^{-\lambda t}\right)\omega^2(t)\hat{q}^2 \qquad (1)$$

The Lagrangian of equation (1) becomes:

$$L = \frac{1}{2}m_0e^{-\lambda t}\left[\dot{q}^2 - \omega^2(t)q^2\right] \qquad (2)$$

and its equation of motion for the classical co-ordinate q and momentum p takes the form

$$\ddot{q}(t) - \lambda\dot{q}(t) + \omega^2(t)q = 0 \qquad (3)$$

The solution of Equation (3) is

$$q(t) = e^{\frac{\lambda}{2}t}\left[Ae^{i\Phi t} + Be^{-i\Phi t}\right], \qquad (4)$$

where $\Phi(t) = \sqrt{\left(\frac{\lambda}{2}\right)^2 - \omega^2(t)}$

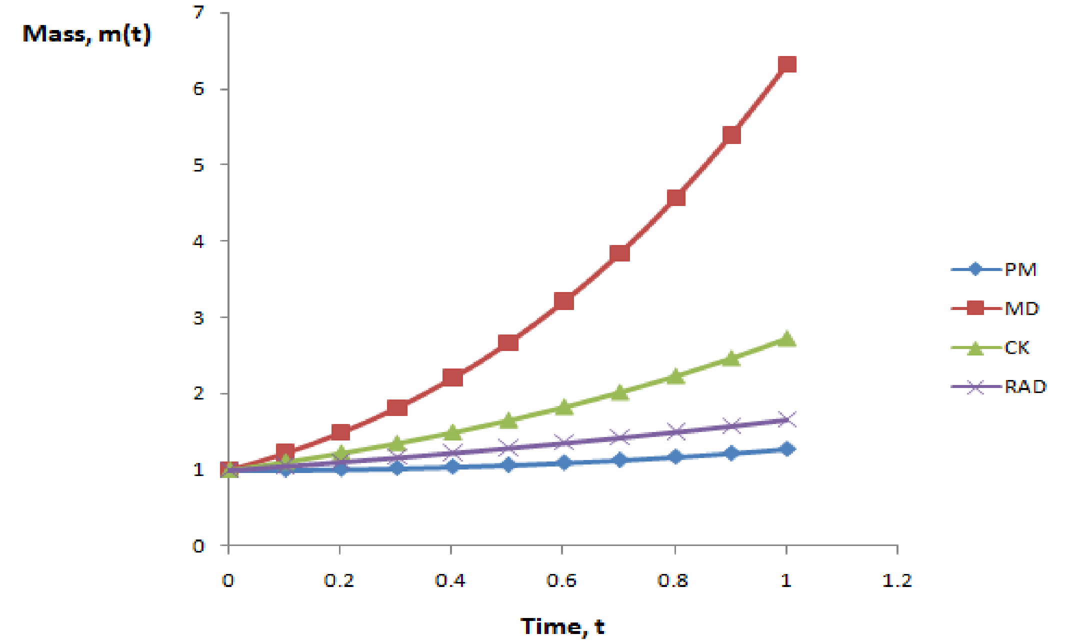

Figure 1: Comparing the masses of four types of non consecutive systems viz the radioactive (RAD), Caldilora-Kanai (C-K), and the Antia's (PM and MD)

The wave function and energy eigenvalue are, respectively, given as:

$$\psi_n(q,t) = \left(\frac{\left(m/\hbar \right)^{1/2}}{\pi^{1/2} 2^n n!} \right) H_n \left[e^{-\lambda t} \left(\frac{m\omega}{\hbar} \right)^{\frac{1}{2}} q \right]$$

$$Exp\left[-\left(\frac{m\omega}{2\hbar} e^{-2\lambda t} x^2 + i\omega \left(n + \frac{1}{2} \right) t e^{\lambda t} \right) \right]. \quad (5)$$

and
$$E_n = \left(n + \frac{1}{2} \right) \hbar \omega e^{\lambda t} \quad (6)$$

We summarized the general solution of Equation (3) for the Over-Damped (OD), Under-Damped (UD) and Critically Damped (CD), respectively, as:

$$q(t) = e^{\frac{\lambda}{2}t} \left[ACosh\Phi t + BSinh\Phi t \right], \quad (7)$$

$$q(t) = e^{\frac{\lambda}{2}t} \left[ACos\Phi t + BSin\Phi t \right], \quad (8)$$

and
$$q(t) = e^{\frac{\lambda}{2}t} \left[A + Bt \right], \quad (9)$$

INVESTIGATION OF THE UNDER-DAMPED, OVER-DAMDED AND CRITICALLY DAMPED OSCILLATIONS FOR THE RADIOACTIVE MODEL

For the pulsating mass model, subjecting q(t) to continuity condition (Antia (2010), *q(0)=1 and* $\dot{q}(0) = i\Omega$, we obtain the arbitrary constant A and B as:

$$\left.\begin{aligned} A &= 1 \\ B &= i \end{aligned}\right\} \qquad (10)$$

and the classical trajectory becomes:

$$q(t) = e^{\frac{-\gamma t}{2}\tanh\frac{\gamma t}{2}}\left[Cos\Omega t + iSin\Omega t\right], \qquad (11)$$

where

$$\Omega^2 = \omega^2(t) - \frac{\gamma}{4}\tanh^2\frac{\gamma t}{2}$$

Following Lewis and Riesendeld (1969), we can now introduce a pair of operators first order in both position and momentum operators (Kim and Page, 2001), Kim *et al*, 2003)):

$$\hat{a}(t) = i\left[\varepsilon^*(t)\hat{p} - \dot{\varepsilon}^*(t)\hat{q}\right] \qquad (12)$$
$$\hat{a}^+(t) = i\left[\varepsilon(t)\hat{p} - \dot{\varepsilon}(t)\hat{q}\right]$$

Required to satisfy the quantum Liouville-Von Neumann equation defined as:

$$i\hbar\frac{\partial}{\partial t}\hat{a}(t) + \left[\hat{a}(t), \hat{H}(t)\right] = 0$$
$$i\hbar\frac{\partial}{\partial t}\hat{a}^+(t) + \left[\hat{a}^+(t), \hat{H}(t)\right] = 0 \qquad (13)$$

where $\varepsilon(t)$ in equation (12) must satisfies the corresponding classical damped equation.

The operator in equation (12) and its Hermitian conjugate satisfy at any time t the boson commutation relation, and $\varepsilon(t)$ must also satisfy the Wronskian condition (Kim, 2004).

For Antia (2010) model of pulsating mass, we have

$$e^{\frac{d}{d\beta}\ln Cosh^2\beta\frac{\gamma t}{2}}\left[\varepsilon^*(t)\varepsilon(t) - \dot{\varepsilon}(t)\varepsilon^*\hat{q}\right] = i \qquad (14)$$

The number operator defined by (Ikot *et al*, 2008)

$$\hat{N}(t) = \hat{a}^t(t)\hat{a}(t) \qquad (15)$$

also satisfies equation (15), such that each number state

$$\hat{N}(t)|n,t\rangle = n|n,t\rangle \qquad (16)$$

is also an exact quantum state of the time dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \hat{H}(t)\psi(x,t) \qquad (17)$$

The quantum dispersion coordinate is obtained as

$$\langle\hat{q}^2\rangle = \frac{\hbar}{2m_0\Omega}\varepsilon^*(t)\varepsilon(t) = \frac{\hbar}{2m_0\Omega}e^{-\gamma t\tanh\frac{\gamma t}{2}} \qquad (18)$$

And the uncertainty in momentum is given as

$$\langle\hat{p}^2\rangle = \frac{\hbar}{2m_0\Omega}m(t)^2\dot{\varepsilon}^*(t)\dot{\varepsilon}(t) = \frac{\hbar}{2}m_0\Omega e^{\gamma t\tanh\frac{\gamma t}{2}}$$

$$\text{x}\left[1+\frac{1}{\Omega^2}\left(-\frac{\gamma}{2}\tanh\frac{\gamma t}{2}-\frac{\gamma^2 t}{2}Sech^2\frac{\gamma t}{2}\right)^2\right] \quad (19)$$

where $m(t)$ is the reduced mass of the oscillator, which is defined as (Ikot *et al*. 2010)

$$m(t) = m_0 e^{\frac{d}{d\beta}\ln Cosh^{2\beta}\frac{\gamma t}{2}} \quad (20)$$

and β is a variable parameter that takes values 1,2,3,…;n. By setting $\beta = 1$ equation (17) is obtained.

The generalized uncertainty relation has the value:

$$(\Delta q\Delta p)^2 = \frac{\hbar^2}{4}\left[1+\left(\frac{\gamma}{2\Omega}\right)^2\left(\tanh\frac{\gamma t}{2}+\frac{\gamma t}{2}Sech^2\frac{\gamma t}{2}\right)^2\right] \quad (21)$$

Equation (21) is a generalized uncertainty relation and it satisfies the Heisenberg uncertainty relation. The product of equations (18) and (19) gives a generalized Heisenberg relation which reduces to the exact when the damping coefficient γ is set to zero (Antia 2010).
Similar results for the radioactive model, with λ as the damping coefficient, are obtained as:

(i) FOR UNDER-DAMPED OSCILLATION

$$q(t) = e^{\frac{\lambda}{2}t}\left[Cos\Phi t + iSin\Phi t\right],$$

$$\langle\hat{q}^2\rangle = \frac{\hbar}{2m_0\Phi}e^{\lambda t}\left[1+\frac{\lambda^2}{4\Phi^2}\sin^2\Phi t+\frac{\lambda}{2\Phi}\sin 2\Phi t\right] \quad (22)$$

$$\langle\hat{p}^2\rangle = \frac{\hbar}{2}m_0\Phi e^{-\lambda t}\left[1+\frac{\lambda}{\Phi}\sin 2\Phi t-\frac{\lambda^2}{\Phi^2}\sin^2\Phi t+\frac{\lambda^2}{\Phi^2}\left(\frac{\lambda}{2\Phi}\cos 2\Phi t-\frac{\lambda}{\Phi}\sin 2\Phi t-1\right)\right.$$
$$\left.+\frac{\lambda}{\Phi^2}\left(\frac{\lambda}{\Phi}\sin 2\Phi t-2\cos 2\Phi t\right)\right] \quad (23)$$

Then the corresponding generalized Heisenberg relation becomes

$$(\Delta q\Delta p)^2 = \frac{\hbar^2}{4}\left[1+\delta(t)\right] \quad (24)$$

Where

$$\delta(t) = \left(1+\frac{\lambda^2}{\Phi^2}\sin^2\Phi t+\frac{\lambda}{\Phi}\sin 2\Phi t\right)\left[1+\frac{\lambda^2}{\Phi^2}\sin^2\Phi t+\frac{\lambda^2}{\Phi^2}\left(\frac{\lambda}{2\Phi}\cos 2\Phi t-1\right)\right.$$
$$\left.+\frac{\lambda}{\Phi^2}(2\cos 2\Phi t)+\frac{\lambda}{\Phi}\sin 2\Phi t\right]$$

Hence

$$\Delta q \Delta p \geq \frac{\hbar}{2} \qquad (25)$$

which is in compliance with Heisenberg uncertainty principle.

(ii) FOR OVER-DAMPED OSCILLATION

The over-damping occurs when λ > ω. The solution, with boundary conditions imposed, will then take the form:

$$q(t) = e^{\frac{\lambda}{2}t}\left[Cosh\Phi t + \left(i - \frac{\lambda}{2\Phi}\right)Sinh\Phi t\right], \qquad (26)$$

The uncertainty in the coordinate is obtained as

$$\langle \hat{q}^2 \rangle = \frac{\hbar}{2m_0\Phi} e^{\lambda t}\left[\cosh 2\Phi t + \frac{\lambda}{\Phi}\sinh 2\Phi t + \frac{\lambda^2}{\Phi^2}\sinh^2 \Phi t\right] \qquad (27)$$

Similarly, the uncertainty in the momentum is

$$\langle \hat{p}^2 \rangle = \frac{\hbar}{2} m_0 \Phi e^{-\lambda t}\left[\cosh 2\Phi t\left(1 - 3\frac{\lambda^3}{\Phi^3}\right) - \frac{\lambda}{\Phi}\sinh 2\Phi t\left\{2 + \frac{\lambda}{\Phi} + 2\frac{\lambda^2}{\Phi^2} + \frac{\lambda^3}{\Phi^3}\right\}\right] \qquad (28)$$

Hence

$$(\Delta q \Delta p)^2 = \frac{\hbar^2}{4}\left[\left\{\cosh 2\Phi t + \frac{\lambda}{\Phi}\sinh 2\Phi t + \frac{\lambda^2}{\Phi^2}\sinh^2 \Phi t\right\}\left\{\cosh 2\Phi t\left(1 - 3\frac{\lambda^3}{\Phi^3}\right) - \frac{\lambda}{\Phi}\sinh 2\Phi t\left(2 + \frac{\lambda}{\Phi} + 2\frac{\lambda^2}{\Phi^2} + \frac{\lambda^3}{\Phi^3}\right)\right\}\right] \qquad (29)$$

(iii) FOR CRITICALLY DAMPED OSCILLATION

Critically damped oscillation occurs when $\frac{\lambda^2}{4} = \omega^2$

Using the continuity equation,

$$q(t) = 1 - \frac{i\Phi}{\lambda}\left(1 - e^{\lambda t}\right)$$

Thus

$$\langle \hat{q}^2 \rangle = \frac{\hbar}{2m_0\Phi}\left[1 + \frac{\Phi^2}{\lambda^2}\left(1 - e^{\lambda t}\right)^2\right] \qquad (30)$$

and

$$\langle \hat{p}^2 \rangle = \frac{\hbar}{2} m_0 \Phi^3 e^{2\lambda t} \qquad (31)$$

$$(\Delta q \Delta p)^2 = \frac{\hbar^2}{4} \lambda^2 \Phi^2 e^{2\lambda t} \left[1 + \Phi^2 \left(1 - e^{\lambda t}\right)^2 \right] \qquad (32)$$

CONCLUSION

Since equations 29 and 32, which are the results of the over-damped and critically damped, do not necessarily lead to $\Delta q \Delta p \geq \frac{\hbar}{2}$, then it is only the under-damped oscillation of the radioactive model that shows no violation of Heisenberg uncertainty relation, which is the basic Quantum Mechanical principle.

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