AN ESTIMATE NECESSARY FOR SOLVING NON-HOMOGENEOUS SEMI-LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT: In this paper, we state and prove some major assumptions and estimate necessary for solving any non-homogeneous semi-linear differential equations. Using the idea of boundedness of the first partial derivative of a function that is Lipschitz's continuous with respect to the displacement function "u", we gave an estimate that $\left|(f_u u', u'')\right| \leq c \left(\frac{d}{dt}\left\|u'\right\|^2_{L^2(\Omega)}\right)$, $c > 0$.

INTRODUCTION

To solve any non-homogeneous differential equation, we determine the well-posedness of the problem. The idea of a classical solution, in which case the function is required to have certain degree of smoothness or being k-times continuously differentiable depending on the order of the equation had been considerably weakened by the introduction of a generalize solution. It is often the case that in establishing the existence of a generalized solution depends upon rather simple a priori estimates with ideas of functional analysis while the regularity of such weak solution rests upon many intricate calculus estimates. (Evans 1998).
Consider the differential equation of the form;

$$\sum_{|\propto|=k} a_{\propto}(x)D^{\propto} u + a_0 \left(D^{k-1} u, \dots, Du, u, x\right) = 0 \quad 1$$

This in a more precise representation may be any of these equations:

$$u_t(x,t) - \Delta u(x,t) = f(x,t,u) \quad x \in \Omega, t > 0 \quad 2$$

$$u_{tt}(x,t) - \sum_{i,j} a^{i,j}(x,t)u_{xixj} = f(x,t,u) \quad in \ \Omega \text{x}(0,\infty) \quad 3$$

with their prescribed data.

The solvability of these problems is the discovery of ways to obtaining the unknown function "u", if it exists, that must satisfy the given equation. This leads to the idea of investigating the existence and uniqueness of the solution, which theoretically is the notion of well-posedness of the problem.Various techniques had been employed by Mathematicians to solve differential equations of types (2) and (3), as solutions can only be viewed depending on the type, (Aliev and Tahir 2005, El Mahjoub 1998, Fairweather and Lopez 1996). The generalized (weak) solution is to remove the restrictions of a classical solution which often pose difficulties due to the non-differentiability of various functions but certain functional analysis deductions and function spaces requirements must be logically applied. This solution is based on the theory of distribution and as such, Sobolev's Space becomes the best Space for consideration in that it answers the questions about the boundedness of derivatives of functions.

Definition 1. A distribution or generalized function is a continuous linear map u : $C_0^\infty \to R$ such that $\varphi \to (u, \varphi)$ in the sense that if $\varphi_n \to \varphi \ in \ C_0^\infty, then \ (u, \varphi_n) \to (u, \varphi)$. (Renardy and Robert 1993).

Where φ is usually a test function from the space of solution considered with compact support. We now state and prove some necessary assumptions towards solving such problems and give the estimate.

Necessary Assumptions

Most of the assumptions are based on Sobolev's Space properties, indicating the boundedness of the function and its derivatives.

Definition 2: (Sobolev's Space) (Folland, 1976)
In a fixed interval $1 \leq p \leq \infty$, let s be a non-negative integer, the space written $W^{s,p}(\Omega)$ consists of all locally summable functions $u: \Omega \rightarrow R$ such that for each multi-index $\propto, |\propto| \leq s, D^{\propto}u \in L^p(\Omega)$.

Assumption 1: Let $f(x,t,u)$ be a real continuously differentiable function that is Lipschitzian in "u", then there exists $C_1 > 0$ such that $|f_u(x,t,u)| \leq C_1$. This assumption follows from the fact that if "f" is a function that is locally Lipschitzian, then it is continuous and it is shown also that its first partial derivatives are continuous and bounded, (Chidume 1995)

Assumption 2: Let $f(x,t,u)$ be Lipschitz continuous in "u", then in a relative compact region Ω, we have that $|f_u u'|_{L^2(\Omega)} \leq C_1\|u'\|_{L^2(\Omega)}$, $C_1 \geq 0$.

Proof: Consider any two arbitrary points in the region (x,t,u_1) and (x,t,u_2) , there exists a positive integer "L" ∋ $|f(x,t,u_1) - f(x,t,u_2)| \leq L|u_1 - u_2|$

Given $\varepsilon > 0\, and\, \delta > 0, where\, |u_1 - u_2| < \delta$, we obtain the fact that $|f(x,t,u_1) - f(x,t,u_2)| < \epsilon$

Thus, f is continuous and therefore bounded. Following Sobolev's inequality of the definition 2 where $f \in L^2(0,T;L^2(\Omega))\, and\, Df \in L^2(0,T;L^2(\Omega)), it\, implies\, that$ "f_u" does exists and by applying the mean value theorem within a stipulated interval, we have that "f_u" is bounded. Now, let $|f_u u'|$ be majorized by the estimate $|f_u||u'|$ (Oleg, 2003). This expression does exist, when the various a priori estimates in semi-linear or quasi-linear problems will be required.

We write $|f_u u'| \ll |f_u|\|u'\|$ $then$ $|f_u u'|_{L^2(\Omega)} \leq C_1\|u'\|_{L^2(\Omega)}$.

This assumption is necessary and can be extended to higher order derivatives as need arises, depending on the estimates required.

RESULT

Arising from the above assumptions and some calculus estimates that

A: $(u', u'') = \frac{d}{dt}\left(\frac{1}{2}\|u'\|^2_{L^2(\Omega)}\right)$ 4

B: $|(f,u')| \leq \frac{\rho}{\epsilon}\|f\|^2_{L^2(\Omega)} + \epsilon\|u'\|^2_{L^2(\Omega)}$, $\rho > 0$ (Young's inequality, Nasser 2002) 5

We state and prove the following;

Theorem: Let $f(x,t,u)$ be Lipschitz continuous in "u" such that "f_u" exists and bounded in a relative compact region "Ω" with compact support, then we have the estimate
$|f_u u', u''|_{L^2(\Omega)} \leq C\left(\frac{d}{dt}\|u'\|^2_{L^2(\Omega)}\right)$, $C > 0$.

Proof: The estimate follows from assumption 2 above.
Let $|(f_u u', u'')| \ll |(C_1 u', u'')| \leq C(u', u'')$, by the result in A it implies that $\leq C\left(\frac{d}{dt}\|u'\|^2\right)$.

CONCLUSION

This estimate is of great advantage to obtaining a priori estimate for solving any semi or quasi linear problems.

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