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INVESTIGATING THE CHAOTICITY OF HENON-HEILES SYSTEM AND A NON-LINEAR HARMONIC CHAOTIC OSCILLATOR

OKON.I.B* AND POPOOLA. O.O

Department of Physics,
University of lbadan, Nigeria.
**e-mail: aitismen@yahoo.com*

ABSTRACT: This work investigates the chaoticity of Henon-Heiles system and a non-linear harmonic chaotic oscillator in terms of variation in energy (control parameter). The result shows that for Henon-Heiles system, increasing the energy increases the chaotic state of the system. The energy =1/6 is the most chaotic state of the system (Threshold value). However, for non-linear harmonic chaotic oscillator, the system becomes chaotic at energy =10 which is the threshold value and these can be seen in the Poincare section of the phase plot.

INTRODUCTION

Chaos theory describes the behaviour of non-linear dynamical systems i.e. systems whose states evolve with time and are highly sensitive to initial conditions. Sensitivity to initial condition means that an imperceptible change in initial condition will culminate to progressively larger changes in the later state of the system. However, the definition of chaotic system can be encapsulated to be an operational non-linear dynamical system which is bounded and periodic, exponentially sensitive to an initial condition and is also associated with topologically mixing of the orbit (Cvitanovic et al., 2010). The study of chaos becomes paramount and essential because it cuts across almost all discipline. Smale, (1998) discovered the horse shoe map that play an important role in the explanation of chaos occurrence. Ueda, (1985) constructed the chaotic regions of Duffing's equation. Biswas and Harrison, (1986) studied the chaos of light. (Festa et al., 2001) reformulated Lorenz dynamical system in the phase space in terms of first time exit problem. (Feigenbaum, 1978) discovered a universal constant for a function approaching chaos through periodic doubling. The Feigenbaum constant characterized the geometric series of bifurcation parameter approaching to its initial values. The two Hamiltonian systems considered in this work are non-integrable and time independent.

HENON-HEILES SYSTEM

The Henon-Heiles Hamiltonian was introduced in 1964 as a mathematical model of describing the chaotic motion of stars in the galaxy (Cremers and Mielke, 1999). This Hamiltonian describes two dimensional harmonic oscillators with cubic interaction. It is one of the simplest Hamiltonians to display soft chaos in classical mechanics. By increasing the total energy, a transition from integrable to ergodic system is induced. Michel Henon and Carl Heiles also investigated the existence of third integral of motion in a central potential system (Henon and Heiles, 1964).

(Weissten, 2011) showed that the potential for this system in polar-coordinate is given by

$$V(r,\theta) = \frac{1}{2}r^2 + \frac{1}{3}r^3 \sin 3\theta \qquad 1$$

The potential can be transformed from polar coordinate to Cartesian coordinate with the transformations

$$\left.\begin{aligned} \sin\theta &= \frac{y}{\sqrt{x^2+y^2}} \\ \cos\theta &= \frac{x}{\sqrt{x^2+y^2}} \\ \theta &= \tan^{-1}\frac{y}{x} \\ \sin 3\theta &= 3\sin\theta\cos^2\theta - \sin^3\theta \end{aligned}\right\} \qquad 2$$

$$r^2 = x^2 + y^2$$

Substituting equation (2) into (1) gives

$$V(x,y) = \frac{1}{2}\left(x^2+y^2\right) + \frac{1}{3}\left(x^2+y^2\right)^{\frac{3}{2}}\left[\frac{3x^2y}{\left(x^2+y^2\right)^{\frac{1}{2}}\left(x^2+y^2\right)} - \frac{y^3}{\left(x^2+y^2\right)^{\frac{3}{2}}}\right] \qquad 3$$

Simplifying equation (3) gives

$$V(x,y) = \frac{1}{2}\left(x^2+y^2\right) + x^2y - \frac{1}{3}y^3 \qquad 4$$

But the total energy for Henon-Heiles system is given by

$$E = H = V(x,y) + \frac{1}{2}\left(x^2+y^2\right) = K + V \qquad 5$$

$$E = \frac{1}{2}\left(x^2+y^2\right) + \frac{1}{2}\left(x^2+y^2\right) + x^2y - \frac{1}{3}y^2 \text{ This simplifies to}$$

$$E = \frac{1}{2}\left(x^2+y^2+2x^2y-\frac{2}{3}y^3\right) + \frac{1}{2}\left(x^2+y^2\right) \qquad 6$$

Using Hamiltonian's canonical conjugate parameters:

$\dot{x}_1$ = q_1, $\dot{y}_1$ = q_2

$\dot{x}_2$ = p_1, $\dot{y}_2$ = p_2

Where q_1 and q_2 are the positions of the particle and p_1 and p_2 are their respective momentum in the phase space.

Equation (6) then reduces to

$$E = H(p,q) = \frac{1}{2}\left(q_1^2 + q_2^2 + 2q_1^2 - \frac{2}{3}q_2^3\right) + \frac{1}{2}\left(p_1^2 + p_2^2\right) \quad 7$$

Then,

$$H(p,q) = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{q_1^2}{2} + \frac{q_2^2}{2} + q_1^2 q_2 - \tfrac{1}{3} q_2^3 \quad 8$$

Equation (8) is the Hamiltonian function for Henon-Heiles system. In order to investigate the chaoticity of this system, some orbit in the phase space were computed numerically using Runge-Kutta algorithm of fourth order.

NON-LINEAR CHAOTIC HARMONIC OSCILLATOR

The non linear harmonic chaotic oscillator considered in this work has both integrable and non integrable part. The Hamiltonian is quite simple with the coupling constant C. The Hamiltonian for this system is given by

$$H(p,q) = \frac{1}{2}(P_1^2 + P_2^2) + \frac{1}{2}W^2 q_1^2 + \frac{1}{2}w^2 q_2^2 + \frac{1}{2}C q_1^2 q_2^2 \quad 9$$

This system is also investigated numerically using Runge-Kutta algorithm of fourth order.

MATERIALS AND METHOD

Investigating the chaoticity of these two systems involve the following steps.

1. Deriving the corresponding set of differential equations from the Hamilton's canonical equation of the system.
2. Selecting the algorithm for solving these equations.
3. Coding the Fortran programming language.
4. Implementation of the algorithm.
5. Using the data generated by the program to make Poincare plot.

A Fortran programme simulations using fourth order Runge-Kutta with stepsize dt=0.001,and total phase point of 200 were used to generate data for Poincare section for both Hamiltonian systems using different set of initial conditions. The programming code receives two initial conditions (P_2,q_2)for a particular energy, iteration is then carried out with different sets of initial conditions in conjunction with Poincare plot using Microsoft excel. The whole process is carried out for different energies for the two Hamiltonian systems.

RESULTS AND DISCUSSION

The Poincare plot obtained for different energies for both systems is shown in the Figures below. For Henon-Heiles system, the Poincare section for energies 1/12 and 1/8 (Figures 1 and 2) are made up of chaotic seas and islands and the system is not chaotic. But when the energy has been varied and increased to 1/6 (Figure 3), the system becomes chaotic as the particle spread through the entire phase space, hence 1/6 is the threshold value of the energy for Henon-Heiles system. However, for the non-linear harmonic chaotic oscillator, the system becomes chaotic when the energy has been increased to 10. For Henon-Heiles system if the energy is further increased above 1/6, the particles in the potential well will escape through the three saddle points in the potential function. Poincare section is considered by reducing a four dimensional phase space into two with the condition q=0 and p>0.

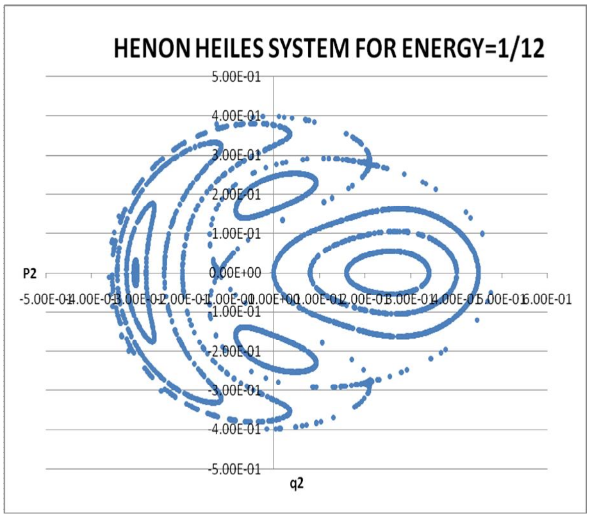

Figure 1

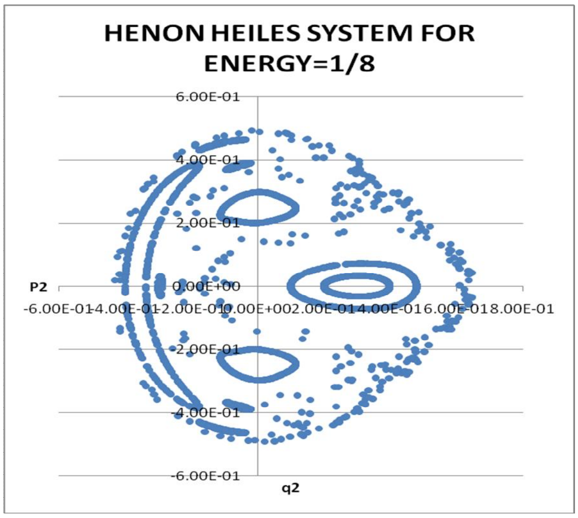

Figure 2

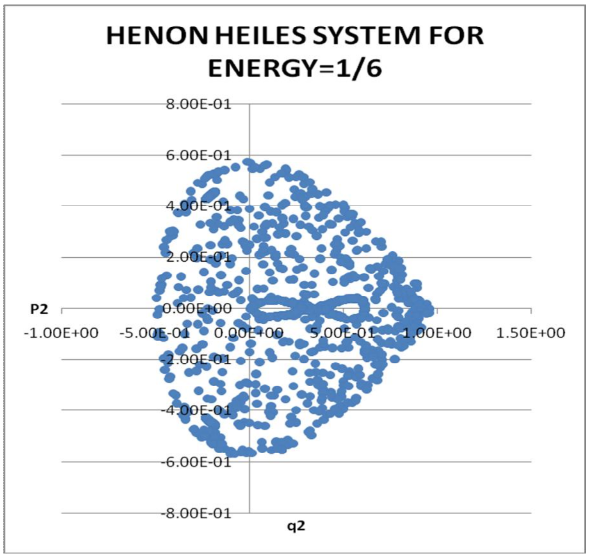

Figure 3

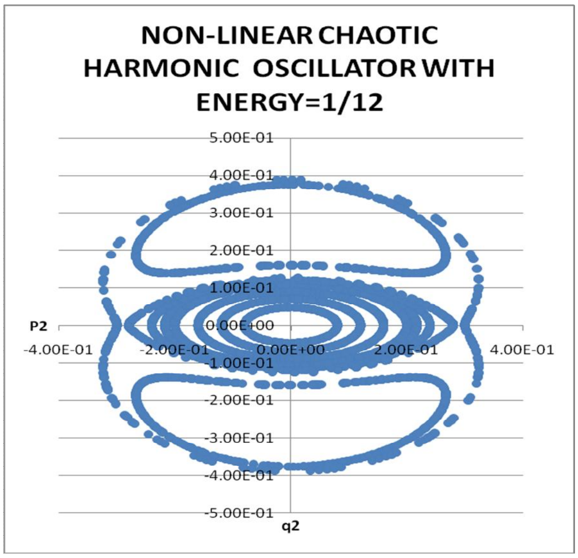

Figure 4

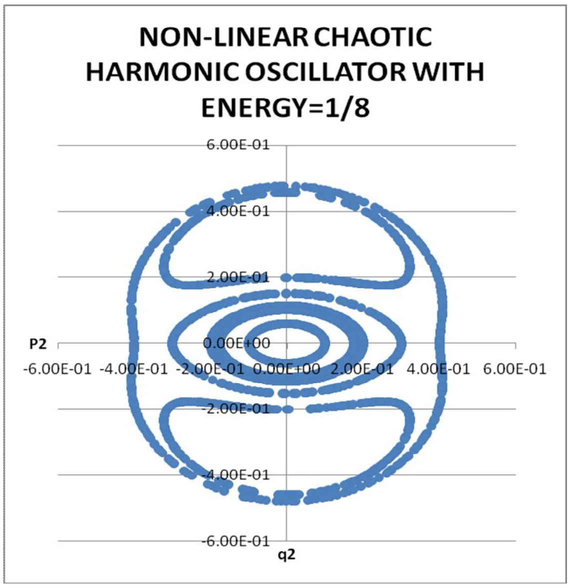

Figure 5

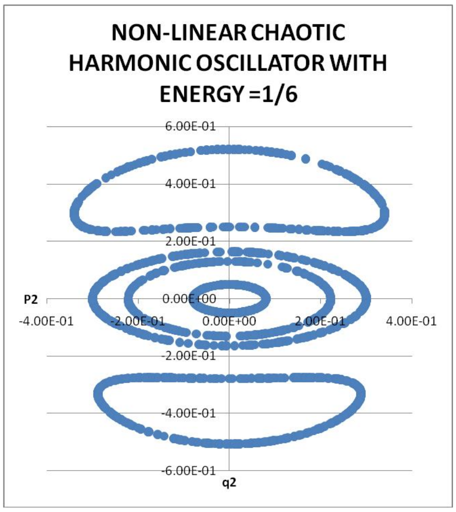

Figure 6

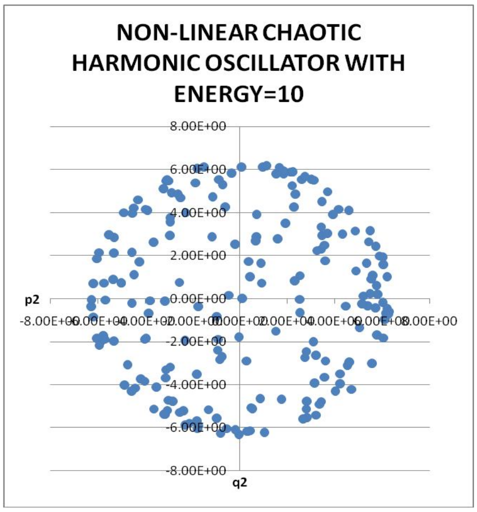

Figure 7

CONCLUSION

From the results, it is obvious that energy plays a major role in determining and investigating the chaoticity of conservative systems. Increasing the energy increases the chaoticity of the system until threshold value is established for both systems.

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