ON (δ, k) - WEAK CONTRACTIONS AND FIXED POINT OF L-LIPSCHITZIAN MAPS

ISSN: 2141 – 3290
www.wojast.com

XAVIER UDO-UTUN
Department of Mathematics and statistics,
University of Uyo, Uyo – Nigeria

ABSTRACT: We have proposed an application of a result by Berinde [2003, 2004] on Picard's iteration of (δ, k)-weak contraction to prove existence of fixed points of a class of L-Lipschitzian operators which is larger than the class of contraction mappings.

INTRODUCTION

Definition 1: Let (X, d) be a metric space and T: $X \to X$ a mapping such that there exist a least constant $L > 0$ with $d(T_x, T_y) \leq Ld(x, y)$ for all $x, y \in X$, then T is called L-Lipschitzian operator.

T is called a contraction for $L < 1$ and for $L = 1$ T is called a nonexpansive operator. A point $p \in X$ is called a fixed point of an operator T if $p = Tp$ and the collection of all fixed points of an operator T is denoted by Fix (T). Fixed points of Lipschitzian maps have been studied by several authors for the classes of operators like nonexpansive maps and strictly pseudicontractive maps, Browder and Petryshym, (1967), Chidume *et al*, (2001), and Igbokwe and Udo-utun (2010). In has been shown that the Mann iteration scheme which is more general than the Picard's iteration scheme does not always converge to a fixed point of Liptschitz Psuedocontractive maps, Chidume and Matangaruda, (2001). It should be noted that salient properties of such operators are dependent on structure of underlying spaces.

It is still open to determine conditions independent of structures of underlying spaces under which an L-Liptschitzian map has fixed point. In this work we shall apply a result of Berinde (2003, 2004) to prove existence of, and convergence of Pacard's iteration scheme to fixed points of a class L-Liptschitzian maps which is not necessarily the class of contractive operators.

Definition 2: Let X be a metric space, $\delta \in (0,1)$ and $k \geq 0$, then a mapping T: $X \to X$ is called (δ, k)-weak contraction (or weak contraction) if an only if

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, T_x), \text{ for all } x, y \in X \tag{1}$$

It is shown in Berinde (2003, 2004) that a lot of well known contractive conditions in literature are special cases of weak contraction condition (1) as it does not required that $\delta + k$ be less than 1 which as assumed in almost all fixed point theorems based on contractive condition which involves displacements of the forms $d(x, y), d(x, T_x), d(y, T_y), d(x, T_y), d(y, T_x)$ (example, Kannan (1968), Zamfirescu, (1992) and Berinde (2003).
Berinde (2004) proved the theorem below:

Theorem 1: Let (X, d) be a complete metric space and T: $X \to X$ a (δ, k)-weak contraction. Then

(1) Fix T) $=\{x \in X : Tx = x\} \neq \phi$.

(2) For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n, n = 0,1,2,...$

(3) The following estimates

$$d(x_{n,} x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), n = 0,1,2... \quad (2)$$

$$d(x_{n,} x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1,} x_n), n = 1,2,... \quad (3)$$

hold, where δ is the constant appearing in (1).

(4) Under the additional condition that there exist $\theta \in (0,1)$ and some $k_1 \geq 0$ such that

$$d(Tx, Ty) \leq \theta . d(x, y) + k_1 . d(x, Tx), \text{ for all } x, y \in X \quad (4)$$

the fixed point x^* is unique and the Picard iteration converges at the rate $d(x_n, x^*) \leq \theta d(x_n, x^*), n \in N$.

It is the purpose of this paper to apply Theorem 1 to prove the existence of fixed points of a certain class of L-Lipschitzian maps not necessarily for $L \leq 1$ by proving that the Picard's Iterations $x_{n+1} = Tx_n, n = 0,1,2,...$ for this class of operators converge to common fixed points of certain (δ, k)-weak contractions.

RESULTS

Theorem 2: Let (X, d) be a complete metric space and T: $X \rightarrow X$ and L-Lipschitzian map. Suppose there exist a (δ, k)-weak contraction S: $X \rightarrow X$ such that

$$d(Tx, Ty) \leq d(Sx, Sy) \text{ for all } x, y \in X' \quad (5)$$

$$d(Tx, Sy) \leq \delta d(x, Sy) \text{ for all } x, y \in X' \quad (6)$$

where X' denotes some neighborhood of some $p \in Fix(S)$. Then T has a fixed point and the Picard iteration converges to a fixed point of T with rates given by (2) and (3).

PROOF

If T is (δ, k)-work contraction then Theorem 1 applies. We shall show that T need not be equal to S by proving that an L-Lipschitzian map T is a (δ, k)-weak contraction if and only if it is a contraction.

If T is a contraction then T, is clearly a (δ, L_1)-weak contraction if and only if it is a contraction.

If T is a contraction then T, is clearly a (δ, k)-weak contraction with $\delta = L$ since $(Tx, Ty) \leq Ld(x, y) + L_1 d(y, Tx)$ for all $L_1 \geq 0$.

On the other hand, suppose T is L-Lipschitzian map and also a (δ, L_1)-weak contraction then $d(Tx, Ty) \leq Ld(x, y)$ and $d(Tx, Ty) \leq \delta d(x, y) + L_1 d(y, Tx)$. Now let $L = \delta + L_0$ then,

$$\begin{aligned} Ld(x, y) &= \delta d(x, y) + L_0 d(x, y) \\ &\leq \delta d(x, y) + L_0 d(y, Tx) + L_0 d(x, Tx) \\ &\leq (\delta + L_0) d(x, y) + 2L_0 d(y, Tx) \end{aligned}$$

But $Ld(x,y) \le \delta d(x,y) + L_1 d(y,Tx)$ since L is an infimum

$$\Rightarrow 2Ld(x,y) \le (2\delta + L_0)d(x,y) + (2L_0 + L_1)d(y,Tx)$$
$$= (L+\delta)d(x,y) + (2L_0 + L_1)d(y,Tx)$$
$$\Rightarrow d(y,Tx) \ge \frac{L-\delta}{2L_0 + L_1} d(x,y)$$

When $y = Tx$ we obtain

$$L - \delta \le 0. \qquad (7)$$

Also, $Ld(x,y) = \delta d(x,y) + (L-\delta)d(x,y)$

$$\le \delta d(x,y) + (L-\delta)d(y,Tx) + (L-\delta)d(x,Tx)$$
$$\le \delta d(x,y) + 2(L-\delta)d(y,Tx) + (L-\delta)d(x,y)$$
$$\Rightarrow 2(L-\delta)d(y,Tx) \ge 0$$
$$\Rightarrow L - \delta \ge 0. \qquad (8)$$

Therefore, from (7) and (8) we obtain $L = \delta$.

The above argument verifies that the class of L Lipschitzian maps satisfying the condition (5) is larger than the class of contractive maps. Next, by Theorem 1 S has a fixed point p and the Picard iteration $z_{n+1} = Sz_n$ converges to is. Applying the condition (5) and (6) we obtain for any $m, n = 0,1,2,3,\ldots$:

$$d(Tx_n, Tx_m) \le d(Sx_n, Sx_m)$$
$$\le \delta d(x_n, x_m) + kd(x_m, Sx_n)$$
$$\le \delta d(x_n, x_m) + kd(Tx_{m-1}, Sx_n)$$
$$\le \delta d(x_n, x_m) + k\delta(x_{m-1}, Sx_n)$$
$$\le \delta d(x_n, p) + \delta(x_m, p) + k\delta(x_{m-1}, p) + k\delta(Sx_n, p)$$
$$\le \delta^n d(x_1, p) + \delta^m (x_1, p) + k\delta^{m-1}(x_1, p) + k\delta(Sx_n, p).$$

On using the fact that $d(Tx, Sy) \le \delta d(x, Sy)$ for all $x, y \in X'$ we conclude that Tx_n is in X' is when ever x_n is in X' and that $d(Tx_n, Tx_m) \to 0$ as $m, n \to \infty$ which proves that the Picard's sequence $\{x_n\}$ given be $x_{n+1} = Tx_n$ converges in X to a fixed points $p \in Fix(S)$. Observe that $k\delta(Sx_n, p) \to 0$ as $n \to \infty$. End of proof.

REFERENCES

Berinde, V. (2003): On the Approximation of Fixed Point of Weak Contraction Mappings, Cambridge J. Math. V (19) 1, 7 – 22.

Berinde, V. (2004): Approximating Fixed Point of Weak Contractions using Picard's Iteration, Nonlinear Analysis Forum V (9)1, 43 – 53.

Browder, S. C. and Pertryshyn, V. (1967): Construction of Fixed Points of Nonlinear Mappings in Hilbert Spaces, J. Math. Anal. Appl. V(20), 197-223.

Chidume, C. E. and Matangaruda, S. A. (2001): An Example of the Mann Iteration Method for Lipschitz Pseudocontractions, Proc. Amer. Math. Soc. V (129) 8, 2359 – 2363.

Chidume, C. E., Abbas, M. and Ali, B. (2007): Convergence of the Mann Algorithm for a Class Pseudocontractive Mappings, Appl. Comput. V(1)1, 1– 6.

Igbokwe, D. I. and Udo-utun, X. A. (2010): An Averaging Implicit Iteration Process for Common Fixed Points of a Finite Family of Asymptoically Pseudocontractive Maps, J. Fixed Point Theory and Applications V (5) 1, 33-48.

Kannan, R. (1996): Some Result on Fixed Points, Bull. Calcutta Math. Soc. V (10) 71 – 76.

Zamfirescu, T. (1992): Fixed Point Theorem in Metric Spaces, Arch. Math. (Basel) 23 292 – 298.