ANALYTICAL SOLUTIONS OF NON-RELATIVISTIC SCHRODINGER EQUATION WITH HADRONIC MIXED POWER- LAW POTENTIALS VIA NIKIFOROV-UVAROV METHOD.

ISSN: 2141 – 3290

ITUEN, E. E., ANTIA, A. D. AND OKON, I. B.
Theoretical Physics Unit
Department of Physics, University of Uyo-Nigeria.
e-mail: enoituenus@gmail .com
and antiacauchy@yahoo.com

ABSTRACT: In this paper, we have employed a mathematical tool of Nikiforov-Uvarov method to study the Schrödinger equation with mixed power-law potentials. The result gives explicitly the energy eigenvalues and corresponding wave function expressed in terms of the Laquerre polynomials. As special cases, the mixed power-law potential reduces to those of Mie type, Kratzer, Kratzer-Fues, Coulomb and Harmonic potentials. The expectation values of $\langle r^{-2}\rangle$ and $\langle r^{-1}\rangle$ for generalized type I mixed power law potential and $\langle r^{2}\rangle$ and $\langle r^{-2}\rangle$ for generalized type II mixed power law potential have also been evaluated using the Hellmann Feynman Theorem.

INTRODUCTION

The exact bound state solutions of the Schrödinger Equation (SE) with some physical potential play a significant role in quantum mechanics. Recently, the exact solutions of SE have attracted a lot of interest. In the non-relativistic quantum mechanics, some physical potentials have been successful for the description of hadrons as bound states of quarks (Lucha *et al*,1991 and Lim, 2005a) when the Hamiltonian is interpreted as an effective Hamiltonian with the coefficient normalized by the relativistic correction. The relationships between the phenomenological potentials and Quantum Chromo-Dynamics (QCD) potentials have attracted a major interest (Bowler, 1999). The relationships between these potentials help in the theoretical modeling of inter-quark potential since it depends on the choice of the potentials (Lim,2004 and Lim, 2005b). In this paper, we take interest in the bound state solution of the non-relativistic Schrödinger equation arising from the relationship between these two potentials.

The mixed-power law potential is defined as (Eichton *et al,* 1980 and Martin,1988)

$$V(r) = Ar^{a} - Br^{-b} - c, \tag{1}$$

where A, B, c, a, b are constants. Relationship between these potentials have recently developed for relating molecular fields which is of high interest in chemical physics (Lim,2005c and Lim,2005d) and for interatomic potentials which is also useful in condensed matter physics(Lim,2005f and Lim,2005f).

In order to find the relationship among the mixed – power-law potential of Eq. (1), we write (Lim, 2005a)

$$V_1(r) = A_1 r^{a_1} - B_1 r^{-b_1} - c_1 \tag{2}$$

and

$$V_2(r) = A_2 r^{a_2} - B_2 r^{-b_2} - c_2 \tag{3}$$

The relationship between (2) and (3) are defined as

$$\frac{d^n V_1(r)}{dr^n} = \frac{d^n V_2(r)}{dr^n}, (n = 0, 1, 2) \tag{4}$$

Substituting Eqs. (2) and (3) into Eq. (4) and solving the resulting equation yields the following relations (Lim,2005a).

$$A_1 = \frac{a_2}{a_1}\left(\frac{a_2+b_1}{a_1+b_1}\right)A_2 r^{(a_1 \quad a_2)} + \frac{b_2}{b_1}\left(\frac{b_1-b_2}{a_1+b_1}\right)B_2 r^{(b_2+a_1)} \tag{5}$$

$$B_1 = \frac{b_2}{b_1}\left(\frac{b_2+a_1}{b_1+a_1}\right)B_2 r^{(b_1-b_2)} + \frac{a_2}{b_1}\left(\frac{a_1-a_2}{b_1+a_1}\right)A_2 r^{(a_2+b_1)} \tag{6}$$

$$C_1 = \left[\frac{a_2 b_1(a_2+b_1) - a_1 a_2(a_1-a_2)}{a_1 b_1(a_1+b_1)} - 1\right]A_2 r^{a_2} - \left[\frac{b_2 a_1(b_2+a_1) - b_1 b_2(b_1-b_2)}{b_1 a_1(b_1+a_1)}\right]B_2 r^{-b_2} + c_2 \tag{7}$$

However, the complete solution of these two potentials $V_1(r)$ and $V_2(r)$ are related to $V(r)$. By substituting Eqs. (5 – 7) into Eq. (2) and comparing it with Eq. (1), we obtain the following parameters:

$$A = \alpha_1 - \beta_2 - \delta_1 \tag{8}$$

$$B = \alpha_2 - \beta_1 + \delta_2 \tag{9}$$

$$C = c_2 \tag{10}$$

$$a = a_2 \tag{11}$$

$$b = b_2 \tag{12}$$

where

$$\alpha_1 = \frac{a_2}{a_1}\left(\frac{a_2+b_1}{a_1+b_1}\right)A_2 \tag{13}$$

$$\alpha_2 = \frac{b_2}{b_1}\left(\frac{b_1-b_2}{a_1+b_1}\right)B_2 \tag{14}$$

$$\beta_1 = \frac{b_2}{b_1}\left(\frac{b_2+a_1}{b_1+a_1}\right)B_2 \tag{15}$$

$$\beta_2 = \frac{a_2}{b_1}\left(\frac{a_1-a_2}{b_1+a_1}\right)A_2 \tag{16}$$

$$\delta_1 = \left[\frac{a_2 b_1(a_2+b_1) - a_1 a_2(a_1-a_2)}{a_1 b_1(a_1+b_1)} - 1\right]A_2 \tag{17}$$

$$\delta_2 = \left[\frac{b_2 a_1(b_2+a_1) - b_1 b_2(b_1-b_2)}{b_1 a_1(b_1+a_1)} - 1\right]B_2 \tag{18}$$

Generally, we define the mixed-powerlaw potential of Eq. (1) as

$$V(r) = (\alpha_1 - \beta_2 - \delta_1)r^{a} - (\alpha_2 - \beta_1 + \delta_2)r^{-b} - c_2 \tag{19}$$

We will attempt to solve the bound state solution of the Schrödinger equation with the mixed-power-law potential of Eq. (19) for two cases: (i) the generalized type-I mixed-power-law potential corresponding to $a = -1$, $b = 2$ (ii) the generalized type-II mixed-power-law potential corresponding to $a = 2$, $b = 2$, using the Nikiforov-uvarov method (Nikiforov and Uvarov,1988).

Concept of the NU method

The NU method (Nikiforov and Uvarov, 1988) is based on the solution of a generalized second order linear differential equation with special orthogonal functions (Nikiforov and Uvarov,1988 and Ikot and Akpabio,2010). The Schrödinger equation

$$\psi''(x) + [E - V(r)]\psi(x) = 0 , \tag{20}$$

can be solved by this method. This can be done by transforming this equation into a generalized equation of hypergeometic type with an appropriate $s = s(x)$co-ordinate transformation to give:

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 , \tag{21}$$

Where $\sigma(s)$ and $\bar{\sigma}(s)$ are polynomials of at most second order and $\bar{\tau}(s)$ is a first order polynomial. In order to find the exact solutions to Eq. (21), we set the wave function as

$$\psi(s) = \varphi(s)\chi(s), \tag{22}$$

and on substituting Eq. (22) into Eq. (21), reduces Eq. (21) into hyper-geometric type,

$$\sigma(s)\chi''(s) + \tau(s)\chi'(s) + \lambda\chi(s) = 0 \tag{23}$$

where the wave function $\varphi(s)$ is defined as logarithmic derivative (Nikiforov and Uvarov,1988)

$$\frac{\varphi'}{\varphi} = \frac{\pi(s)}{\sigma(s)} \quad . \tag{24}$$

where $\pi(s)$ is a first- order polynomial function.

The λ in equation (23) satisfies the following second order differential equation

$$\lambda = \lambda_n = -n\frac{d\tau}{ds} - \frac{n(n-1)}{2}\frac{d^2\sigma}{ds^2}, n = 0, 1, 2 \ldots \tag{25}$$

The other part of the wave function $\chi(s)$ is the hypergeometirc-type function whose polynomial solution are given by the Rodriques relation,

$$\chi_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}[\sigma^n(s)\rho(s)] , \tag{26}$$

B_n, is a normalization constant and the weight function $\rho(s)$ must satisfy the condition,

$$\frac{d}{ds}(\sigma(s)\rho(s)) = \tau(s)\rho(s) \tag{27}$$

With $\tau(s) = \bar{\tau}(s) + 2\pi(s)$ (27b)

For the weight function $\rho(s)$ to be satisfied, it is necessary that the classical orthogonal polynomials $\tau(s)$ be equal to zero at some point of an interval (a, b) and the derivative of this interval at $\sigma(s) > 0$ will be negative, that is

$$\frac{d\tau(s)}{ds} < 0 \quad . \tag{28}$$

Therefore, the function $\pi(s)$ and the parameter λ required for the NU method are defined as follows

$$\pi(s) = \frac{\sigma' - \bar{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \bar{\tau}}{2}\right)^2 - \bar{\sigma} + k\sigma} , \tag{29}$$

$$\lambda = k + \pi'(s) \tag{30}$$

In principle, since $\pi(s)$ has to be a polynomial of degree at most one, the expression under the square root sign in Eq.(29) must be square of a polynomial of first degree, which is possible if and only if its discriminant is zero. Thus, the equation for k obtained from the solution of Eq. (29) can be further substituted in Eq.(30). On comparing Eq. (25) and Eq. (30), we obtain the energy eigenvalues equation.

Eigenvalue and Eigenfunction of generalized type – I mixedpower-law potential

The generalized type-I mixed power-law potentials are defined from the mixed-power-law potentials as

$$V_{mpI}(r) - \frac{(\alpha_1 - \beta_2 - \delta_1)}{r} - \frac{(\alpha_2 - \beta_1 + \delta_2)}{r^2} - c_2 \tag{31}$$

for $a = -1$ and $b = 2$. The radial part of the Schrödinger equation for this potential is (Ikot and Akpabio,2010)

$$\left[\frac{-\hbar^2}{2\mu}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+\frac{l(l+1)\hbar^2}{2\mu r^2}+V_{mpI}(r)\right]R_{nl}(r)=E_{nl}R_{nl}(r) \quad (32)$$

where r is the internuclear separation, μ is the reduced mass, n is the principal quantum number, E_{nl} is the energy eigenvalues and $\hbar$ is the Plancks constant. Substituting Eq. (31) into Eq. (32), we get

$$\frac{d^2R_{nl}}{dr^2}+\frac{2}{r}\frac{dR_{nl}(r)}{dr}+\frac{2\mu}{\hbar^2}\left[E_{nl}-\frac{(\alpha_1-\beta_2-\delta_1)}{r}+\frac{(\alpha_2-\beta_1+\delta_2)}{r^2}-c_2-\frac{l(l+1)\hbar^2}{2\mu r^2}\right]R_{nl}(r)=0 \quad (33)$$

The Schrödinger equation simplifies to

$$\frac{d^2R_{nl}}{dr^2}+\frac{2}{r}\frac{dR_{nl}(r)}{dr}+\frac{1}{r^2}[\varepsilon^2r^2-\eta_1^2r+\eta_2^2]R_{nl}(r)=0 \quad (34)$$

where the following dimensionless quantities have been employed:

$$\varepsilon^{\dagger}2 = 2\mu/\hbar^{\dagger}2 \ [E_1nl - c_12 \dashv] \quad (35)$$

$$\eta_1^2=\frac{2\mu}{\hbar^2}(\alpha_1-\beta_2-\delta_1) \quad (36)$$

$$\eta_2^2=\frac{2\mu}{\hbar^2}\left(\alpha_2-\beta_1+\delta_2-\frac{l(l+1)\hbar^2}{2\mu}\right) \quad (37)$$

Now comparing Eq. (21) with Eq. (34), we obtain the following polynomials.

$$\sigma(r)=r\ ,\ \bar{\tau}=2$$
$$\bar{\sigma}(r)=(\varepsilon^2r^2-\eta_1^2r+\eta_2^2) \quad (38)$$

Substituting Eq. (38) into Eq. (29), we find the $\pi(r)$ function as

$$\pi(r)=-\frac{1}{2}\pm\begin{cases} i\varepsilon r+\sqrt{1-4\eta_2^2}, for\ k_+ = -\eta_1^2+i\varepsilon\sqrt{1-4\eta_2^2} \\ i\varepsilon r-\sqrt{1-4\eta_2^2}, for\ k_- = -\eta_1^2-i\varepsilon\sqrt{1-4\eta_2^2}\end{cases} \quad (39)$$

For the polynomial of $\tau=\bar{\tau}+2\pi$ which has a negative derivative, we select,

$$\tau(r)=1-2i\varepsilon r+2\sqrt{1-4\eta_2^2} \quad (40)$$

$$K_-=-\eta_1^2-i\varepsilon\sqrt{1-4\eta_2^2} \quad (41)$$

And $$\pi(r)=-\frac{1}{2}-i\varepsilon r+\sqrt{1-4\eta_2^2} \quad (42)$$

With these selectionand $\lambda=k+\pi'(r)$, we write

$$\lambda=-\eta_1^2-i\varepsilon\left(\sqrt{1-4\eta_2^2}+1\right), \quad (43)$$

Another definition of $\lambda_n=-n\tau'-\frac{n(n-1)\sigma''}{2}$

we have $\lambda_n=2ni\varepsilon$. (44)

Comparing Eq. (43) and Eq. (44), we obtain the exact energy eigen-values of radial part of Schrödinger equation with generalized type-I mixed power-law potentials as

$$E_{nl}=\frac{-\left(\frac{2\mu}{\hbar^2}\right)(\alpha_1-\beta_2-\delta_1)^2}{\left[1+2n+\sqrt{1-\frac{8\mu}{\hbar^2}\left(\alpha_2-\beta_1+\delta_2-\frac{l(l+1)\hbar^2}{2\mu}\right)}\right]^2}+c_2 \quad (45)$$

Having obtained the energy values of the radial Schrödinger equation with generalized type-I mixed power-law potential, let us now consider the corresponding wave function. Using the values of $\sigma(r)$ and $\pi(r)$, we obtain

$$\psi(r) = r^{\frac{\mu}{2}} e^{-i\varepsilon r}, \tag{46}$$

where $\mu = \left(1 - 2\sqrt{1 - 4\eta_2^2}\right)$. The weight function is also calculated from Eq.(27) as

$$\rho(r) - r^{\frac{\mu-1}{2}} e^{-i\varepsilon r}. \tag{47}$$

Now using the Rodrique relation and Eqs (46 – 47), we obtain the radial wave function of the Schrödinger equation with generalized type-I mixed power-law potentials as

$$R_{nl}(r) = N_n r^{\frac{\mu}{2}} e^{-i\varepsilon r} L_n^{\sqrt{1-4\eta_2^2}}(i\varepsilon r), \tag{48}$$

where N_n is the normalization constant and $L_n^{\sqrt{1-4\eta_2^2}}(i\varepsilon r)$ is the associated Laquerre polynomials.

Solution of SE for generalized type -II mixed power-law potential

The generalized type-II mixed power-law potential is obtained from the mixed-power-law potential by setting $a = b = 2$ reads

$$V_{mpII}(r) = (\alpha_1 - \beta_2 - \delta_1) r^2 - \frac{(\alpha_2 - \beta_1 + \delta_2)}{r^2} - c_2, \tag{49}$$

The SE for a particle with $V_{mpII}(r)$ can be reduced in the radial part as

$$\frac{d^2 R_{nl}(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[E_{nl} - (\alpha_1 - \beta_2 - \delta_1) r^2 + \frac{(\alpha_2 - \beta_1 + \delta_2)}{r^2} + c_2 - \frac{l(l+1)\hbar^2}{2\mu r^2}\right] R_{nl}(r) = 0 \tag{50}$$

Setting $\xi = r^2$ in Eq. (50), we get

$$\frac{d^2 R_{nl}(\xi)}{d\xi^2} + \frac{1}{2\xi}\frac{dR_{nl}}{d\xi} + \frac{1}{4\xi^2}\left[-\gamma_1^2 \xi^2 + \varepsilon^2 \xi + \gamma_2^2\right] R_{nl}(\xi) = 0 \tag{51}$$

With $\varepsilon^2 = \frac{2\mu}{\hbar^2}(E_{nl} + c_2)$, $\gamma_1^2 = \frac{2\mu}{\hbar^2}(\alpha_1 - \beta_2 + \delta_1)$

$$\gamma_2^2 = \frac{2\mu}{\hbar^2}\left(\alpha_2 - \beta_1 + \delta_2 - \frac{l(l+1)\hbar^2}{2\mu}\right) \tag{52}$$

Similarly, the following polynomials are obtained from Eq. (51):

$$\tilde{\tau}(\xi) = 1, \quad \sigma(\xi) = 2\xi, \quad \tilde{\sigma}(\xi) = -\gamma_1^2 \xi^2 + \varepsilon^2 \xi + \gamma_2^2, \tag{53}$$

and using Eq. (29), we find the $\pi(\xi)$ as

$$\pi(\xi) = \frac{1}{2} \pm \begin{cases} \gamma_1 \xi + \dfrac{1}{2\sqrt{1 - 4\gamma_2^2}}, for\, k_+ = \dfrac{\varepsilon^2 + \gamma_1\sqrt{1 - 4\gamma_2^2}}{2} \\ \gamma_1 \xi - \dfrac{1}{2\sqrt{1 - 4\gamma_2^2}}, for\, k_- = \dfrac{\varepsilon^2 - \gamma_1\sqrt{1 - 4\gamma_2^2}}{2} \end{cases} \tag{54}$$

Hence making the following choice for the polynomial $\pi(\xi)$ as

$$\pi(\xi) = \frac{1}{2} - \left(\gamma_1 \xi - \frac{1}{2\sqrt{1 - 4\gamma_2^2}}\right), \tag{55}$$

gives the function

$$\tau(\xi) = 2 - 2\left(\gamma_1 \xi - \frac{1}{2\sqrt{1 - 4\gamma_2^2}}\right) \tag{56}$$

With this selection, and $\lambda = k + \pi'$, we have

$$\lambda = \frac{\varepsilon^2 - \gamma_1\sqrt{1-4\gamma_2^2}}{2} - \gamma_1, \tag{57}$$

Another definition of $\lambda_n = -n\tau' - \frac{n(n-1)\sigma''}{2}$, yields

$$\lambda_n = 2n\gamma_1 \tag{58}$$

and solving (57) and (58) yields the energy spectrum of the Schrödinger equation with the generalized type-II mixed power-law potential as

$$E_{nl} = \sqrt{\frac{\hbar^2}{2\mu}(\alpha_1 - \beta_2 + \delta_1)}\left[4\left(n+\frac{1}{2}\right) + \sqrt{1 - \frac{8\mu}{\hbar^2}\left(\alpha_2 - \beta_1 + \delta_2 - \frac{l(l+1)\hbar^2}{2\mu}\right)}\right] - c_2 \tag{59}$$

Let us now find the corresponding wave function; we first determine the weight function from Eq. (27) as $\rho(\xi) = \xi^{(\mu+\varepsilon^2)}e^{v\xi}$, (60)

where $\mu = \sqrt{1-4\gamma_2^2}(4-\gamma_1)$ and $v = 4\gamma_1$. Substituting Eq. (60) into the Rodrique relation, we obtain the wave function in terms of Jacobi polynomials as

$$\chi_n = B_n L_n^{\mu+\varepsilon^2, v}(\xi) \tag{61}$$

The other wave function is obtain as

$$\varphi(\xi) = \xi^{\frac{1}{2}\left(1+2\sqrt{1-4\gamma_2^2}\right)} e^{-\frac{v\xi}{2}} \tag{62}$$

Finally, the radial part of the wave function is got as

$$R_{nl}(r) = N_n r^{\left(1+2\sqrt{1-4\gamma_2^2}\right)} e^{\frac{-\gamma_1 r^2}{2}} L_n^{-(\mu+\varepsilon^2, v)}(\gamma_1 r^2) \tag{63}$$

where N_n is the normalization constant.

Calculation of Expectation values of $\langle r^{-2}\rangle, \langle r^{-1}\rangle$ and the Virial theorem for generalized type-I mixed power law potential

Some useful expectation values $\langle r^{-2}\rangle, \langle r^{-1}\rangle$ and the Virial theorem for the mixed power law potential typed-I in an arbitrary numbers of n and l can be obtained by applying the Hellmann-Feynman theorem (HFT). Assuming that the Hamiltonian $\hat{H}$ for a particular quantum mechanical system is a function of some parameter q and letting $E(q) and\ \Psi(q)$ to be the eigenvalues and eigenfunction of the Hamiltonian $\hat{H}$ respectively. The HFT theorem then states that,

$$\frac{\partial E(q)}{\partial q} = \left\langle \Psi(q) \left| \frac{\partial \hat{H}(q)}{\partial q} \right| \Psi(q) \right\rangle. \tag{64}$$

The effective Hamiltonian for this potential is

$$\hat{H} = \frac{-\hbar^2 d^2}{2\mu dr^2} + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2} + \frac{A}{r} - \frac{B}{r^2} - C_2 \tag{65}$$

To find $\langle r^{-2}\rangle$, we let $q = l$ in Eq. (64), then the expectation values for $\langle r^{-2}\rangle$ is

$$\langle r^{-2}\rangle = \frac{16\mu^2(\alpha_1 - \beta_2 + \delta_1)^2}{\hbar^4\sqrt{1 - \frac{8\mu}{\hbar^2}\left(\alpha_2 - \beta_1 + \delta_2 - \frac{l(l+1)\hbar^2}{2\mu}\right)}\left[1 + 2n + \sqrt{1 - \frac{8\mu}{\hbar^2}\left(\alpha_2 - \beta_1 + \delta_2 - \frac{l(l+1)\hbar^2}{2\mu}\right)}\right]^3} \tag{66}$$

The expectation value for $\langle r^{-1}\rangle$, we let $q = A$

$$\therefore \langle r^{-1}\rangle = \frac{-\left(\frac{4\mu}{\hbar^2}\right)(\alpha_1 - \beta_2 - \delta_1)}{\left[1 + 2n + \sqrt{1 - \frac{8\mu}{\hbar^2}\left(\alpha_2 - \beta_1 + \delta_2 - \frac{l(l+1)\hbar^2}{2\mu}\right)}\right]^2} \tag{67}$$

Now, if we let $q = u$ and using the HFT, we obtain the Virial theorem as

$$\langle\psi(\mu)\left|\frac{\delta H(\mu)}{\delta\mu}\right|\psi(\mu)\rangle = \frac{1}{\mu}\langle H - V\rangle = \frac{\partial E_{n,l}}{\partial\mu} \qquad (68)$$

Where

$$(\partial E_{(n,l)})/\partial\mu = \left[-(2/\hbar^2)(\alpha_1-\beta_2-\delta_1)\right]^2/\left[1+2n+\sqrt{(1-8\mu/\hbar^2(\alpha_2-\beta_1+\delta_2-(l(l+1)\hbar^2)/2\mu))}\right]^2 - ((16\mu/\hbar^4)(\alpha_1-\beta_2-\delta_1)^2(\alpha_2-\beta_1+\delta_2))/(\sqrt{(1-8\mu/\hbar^2(\alpha_2-\beta_1+\delta_2-} \qquad (69)$$

$$-\langle H - V\rangle = E_{n,l}(1+\mathcal{G}) - (1+\mathcal{G})C_2$$

Where

$$\mathcal{G} = \frac{\left(\frac{8\mu}{\hbar^2}\right)(\alpha_2-\beta_1+\delta_2)}{\sqrt{1-\frac{8\mu}{\hbar^2}\left(\alpha_2-\beta_1+\delta_2-\frac{l(l+1)\hbar^2}{2\mu}\right)}\left[1+2n+\sqrt{1-\frac{8\mu}{\hbar^2}\left(\alpha_2-\beta_1+\delta_2-\frac{l(l+1)\hbar^2}{2\mu}\right)}\right]}$$

Since $H = \langle T\rangle + \langle V\rangle = E_{n,l}$

Then the Virialtheorem is obtained as

$$-(2+\mathcal{G})\langle T\rangle = (1+\mathcal{G})\langle V\rangle - (1+\mathcal{G})C_2 \qquad (70)$$

Calculation of Expectation values of $\langle r^{-2}\rangle, \langle r^2\rangle$ and the Virial theorem for generalized type-II mixed power law potential

Hamiltonian of this potential is defined as

$$H = \frac{-\hbar^2 d^2}{2\mu dr^2} + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2} + (\alpha_1-\beta_2-\delta_1)r^2 - \frac{(\alpha_2-\beta_1+\delta_2)}{r^2} - C_2 \qquad (71)$$

To find $\langle r^{-2}\rangle$, we let $q = l$

$$\langle r^{-2}\rangle = \frac{\left(\frac{4\mu}{\hbar^2}\right)\sqrt{\frac{\hbar^2}{2\mu}(\alpha_1-\beta_2-\delta_1)}}{\sqrt{1-\frac{8\mu}{\hbar^2}\left(\alpha_2-\beta_1+\delta_2-\frac{l(l+1)\hbar^2}{2\mu}\right)}} \qquad (72)$$

We can now find for $\langle r^2\rangle$, so we let $q = A$

$$\langle r^2\rangle = \frac{\frac{\hbar^2}{4\mu}}{\sqrt{\frac{\hbar^2}{2\mu}(\alpha_{1-\beta_2}-\delta_1)}}\left[4\left(n+\frac{1}{2}\right)+\sqrt{1-\frac{8\mu}{\hbar^2}\left(\alpha_2-\beta_1+\delta_2-\frac{l(l+1)\hbar^2}{2\mu}\right)}\right] \qquad (73)$$

Now the Virial theorem, can be evaluated by setting $q = \mu$. Thus, we have

$$\langle\psi(\mu)\left|\frac{\partial H(\mu)}{\partial\mu}\right|\psi(\mu)\rangle = \frac{-1}{\mu}\langle H - V\rangle = \frac{\partial E_{n,l}}{\partial\mu}, \qquad (74)$$

but $$\left[\partial E\right]_{(n,l)}/\partial\mu = -1/2\mu\sqrt{(\hbar^2/2\mu(\alpha_1\left[-\beta\right]_2-\delta_1))}\left[4(n+1/2)+\sqrt{(1-8\mu/\hbar^2(\alpha_2\left[-\beta\right]_1\left[+\delta\right]_2-l(l+1)\left[\hbar\hbar\right]^2/2\mu))}\right] - \qquad ((4/\hbar^2)\sqrt{((\alpha_1-\beta_2-\delta_1))}(\alpha_2-\beta_1\left[+\delta\right. \qquad (75)$$

Using Eqs (74) and (75), we obtain

$$-\langle H - V\rangle = -\left(E_{n,l}+c_2\right) - \Lambda\left(E_{n,l}+c_2\right) \qquad (76)$$

Where

$$\Lambda = \frac{\left(\frac{4\mu}{\hbar^2}\right)(\alpha_2-\beta_1+\delta_2)}{\sqrt{1-\frac{8\mu}{\hbar^2}\left(\alpha_2-\beta_1+\delta_2-\frac{l(l+1)\hbar^2}{2\mu}\right)}\left[4\left(n+\frac{1}{2}\right)+\sqrt{1-\frac{8\mu}{\hbar^2}\left(\alpha_2-\beta_1+\delta_2-\frac{l(l+1)\hbar^2}{2\mu}\right)}\right]} \qquad (77)$$

Since $H = \langle T\rangle + \langle V\rangle = E_{n,l}$, thus the Virial theorem is obtained as

$$-\Lambda\langle T\rangle = (1+\Lambda)\langle V\rangle + (1+\Lambda)c_2 \tag{78}$$

RESULTS AND DISCUSSION

Generalized Type-I

The mixed power-law potential of type-I of Eq. (31) for $a = 1$, and $b = 2$ is a generalization of four well-known potentials comprises of (i) Mie-type potential (ii) Kratzer potential (iii) Kratzer-Fues potential and (iv) the Coulomb potentials. If we make appropriate choice of parameters in equation (19), we can obtain these four known potentials.

Mie-type potential

Setting $\delta_1 = \beta_2 = 0$, $\alpha_1 = -V_0 a$, $\alpha_2 - \frac{-V_0 a^2}{2}$ $and\ c_2 - 0$,we obtain the Mie-potential as

$$V_{mP}(r) = V_0\left[\frac{1}{2}\left(\frac{a}{r}\right)^2 - \left(\frac{a}{r}\right)\right] \tag{79}$$

The corresponding energy equation and wave function for Eq. (79) are obtained from Eq. (45) and Eq. (48) as

$$E_{nl}^{Mp} = -\frac{2\mu\frac{V_0^2}{\hbar^2}a^2}{\left[1+2n+\sqrt{1+\frac{8\mu}{\hbar^2}\left(\frac{V_0 a^2}{2}+l(l+1)\frac{\hbar^2}{2\mu}\right)}\right]^2} \tag{80}$$

$$R_{nl}(r) = N_n r^{\frac{\mu'}{2}} e^{-i\varepsilon' r} L_n^{\sqrt{1-4\eta_2'^2}} \tag{81}$$

where

$$\mu' = 1 - 2\sqrt{1+\frac{4\mu V_0 a^2}{\hbar^2}+4l(l+1)}\ , \tag{82}$$

$$\eta_2'^2 = -\frac{2\mu}{\hbar^2}\left(\frac{l(l+1)\hbar^2}{2\mu}+\frac{V_0 a^2}{2}\right),\qquad \varepsilon'^2 = \frac{2\mu}{\hbar^2}E_{nl}^{Mp}\ , \tag{83}$$

Kratzer Potential

The Kratzer potential can also be obtained from the generalized mixed power-law potential of type-I. If $\beta_1 = \beta_2 = \delta_1 = \delta_2 = 0$, $\alpha_1 = ka_0$, $\alpha_2 = -ka_0{}^2$ and $c_2 = k$, we obtain the Kratzer potential as (Ikhdair and Sever, 2008)

$$V_{kp}(r) = \frac{ka_0}{r} + \frac{ka_0{}^2}{r^2} + k\ , \tag{84}$$

where k is the interaction energy between two atoms in a molecular system at a distance a_0. The energy eigenvalues of Kratzer potential is obtained from Eq. (45) as

$$E_{nl}^{Kp} = \frac{K - \frac{2\mu k^2 a^2}{\hbar^2}}{\left[1+2n+\sqrt{1+\frac{8ka^2\mu}{\hbar^2}+4l(l+1)}\right]^2} \tag{85}$$

and the radial wave function of Eq. (48) reduces to those of Kratzer potential when $\alpha_1 = ka$ and $\alpha_2 = -ka^2$ and $\beta_1 = \beta_2 = \delta_1 = \delta_2 = 0$.

Kratzer-Fues Potential

The Kratzer-Fue potential is obtain from the generalized mixed power-law potential of type-I by setting $c_2 - 0$,$\alpha_1 - ka$,$\delta_2 = ka^2$, $\beta_1 - \beta_2 - \alpha_2 - \delta_1 - 0$

$$V_{KF}(r) = \frac{ka}{r} - \frac{ka^2}{r^2}\ , \tag{86}$$

where $k = D$ is the interaction energy between two atoms in a molecular systems.The energy eigenvalues obtained in Eq. (45) can be shown to be in agreement with those of the Kratzer-Fue potential by setting the following parameters as

$$E_n^{KF} = \frac{-\frac{2\mu k^2 a^2}{\hbar^2}}{\left[1 + 2n + \sqrt{1 + \frac{8ka^2\mu}{\hbar^2} + 4l(l+1)}\right]^2} \qquad (87)$$

and its radial wave function reduces to Kratzer-Fues when $c = 0$, $\alpha_1 = ka$, $\delta_2 = ka^2$, $\beta_1 = \beta_2 = \alpha_2 = \delta_1 = 0$

Coulomb Potential

For $\alpha_1 = A, \alpha_2 = 0$, $\beta_1 = \beta_2 = \delta_2 = \delta_1 = 0 \; and \; c_2 = 0$, the generalized mixed power-law potential of type-I,reduces to the standard Coulomb potential.

$$V_{cou}(r) = \frac{A}{r}\,, \qquad (88)$$

The corresponding energy eigen values is

$$E_{nl}^{cou} = \frac{-\frac{2\mu A^2}{\hbar^2}}{\left[1 + 2n + \sqrt{1 + 4l(l+1)}\right]^2} \qquad (89)$$

and substituting these parameters into Eq.(48) gives the required radial wave function .

Generalized Type-II

The generalized mixed power-law of type-II of Eq. (19) for the case $a = b = 2$ is also a generalization ofa well-known potential: Harmonic potential . By suitable adjustment of the parameters in the type- II generalized mixed power-law potential of Eq. (19),we can obtain the harmonic potential .

Harmonic Oscillator

The harmonic potential is obtain from this generalized type-II power-law potential by setting $c_2 = 0, \alpha_2 = 0$, $\beta_1 = \beta_2 = \delta_2 = \delta_1 = 0$ and $\alpha_1 = \frac{k}{2}$

$$V_{HP}(r) = \frac{1}{2}kr^2\,, \qquad (90)$$

The energy eigen values are obtained from Eq. (59) as

$$E_{nl}^{H} = \sqrt{\frac{\hbar^2 k}{4\mu}}\left[4\left(n + \frac{1}{2}\right) + \sqrt{1 + 4l(l+1)}\right] \qquad (91)$$

The corresponding radial wave function is also obtained as

$$R_{nl}(r) = N_n r^{\left(1+2\sqrt{(1+4l(l+1))}\right)} e^{-\frac{1}{2}\sqrt{\frac{\mu k r^2}{\hbar^2}}} L_n^{(\mu' + \varepsilon'^2, v')}\left(\sqrt{\frac{\mu k}{\hbar^2}}\, r^2\right), \qquad (92)$$

$$where \; \mu' = \sqrt{1 + 4l(l+1)}\left(4 - \sqrt{\frac{\mu k}{\hbar^2}}\right), v' = 4\sqrt{\frac{\mu k}{\hbar^2}}, \varepsilon'^2 = \frac{2\mu}{\hbar^2}E_{nl}^{H}\,,$$

CONCLUSION

The Hadronic mixed power-law potential has been studied under non relativistic Schroedinger equation in the framework of NikiforovUvarov method with appropriate parameters. This potential is of two types namely: the generalized type-I mixed power-law potential and generalized type-II mixed power-law potential as shown in equations (31) and (49) respectatively. The limiting cases of these potentials reduce to some well known potentials such as Mie-type , Kratzer , KratzerFues , Coulomb and Harmonic potentials. These potentials have some applications in Atomic and Molecular Physics. We have also evaluated expectation values for the two systems using the Virial Theorem. The energy eigenvalues and the wave functions of these potentials, expressed in Laguerre Polynomials, have been obtained accordingly.

ACKNOWLEDGEMENT

The authors are indebted to Dr. Akpan N. Ikot of the Theoretical Physics Unit , Department of Physics, University of Uyo, Nigeria for enlightening, fruitful discussions and invaluable help in literature search.

Bowler, K. C. (1999). Why QCD needs high Performance Computing. *Comput. Phys. Common,* 121/122, 512.

Eichton, E., Gottfried, K., Kinoshita, T., Lane, K. D. & Yan, T. M. (1980). Spectrum of Charmed Quark – Antiquark Bound States. *Phys. Rev.* D21, 203.

Ikhdair, S. and Sever, R. (2008). Exact Solvable Effective Mess D-dimensional Schrodinger equation for Pseudoharmonic and Modified Kratzer problem. *Cent. Euro. J. Phys.* 6, 607.

Ikot, A. N. and Akpabio, L. E. (2010). Approximate Solution of the Schrodinger equation with Rosen-Morse Potential including the Centrifugal Term. *App Phys. Res.* 2 (2), 202.

Lim, T. C. (2004). Application of Maclawin series in Relating Interatomic Potential Functions. *J. Math. Chem.* 36, 147.

Lim, T. C. (2005a). Approximate Relationship between the Generalised Morse and the Extended – Rydberg Potential Energy Functions. *Acta Chim Slov*, 52, 149-152.

Lim, T. C. (2005b). Parametric Relationships Among Some Phenomenological Non-Relativistic Hadromic Potentials. EJTP, 8, 13.

Lim, T. C. (2005c). Polynomial Forms of Typical Interatomic Potential Functions. *J. Math. Chem.,* 38, 495.

Lim, T. C. (2005d). Scaling Factors between the 2-Body Energy of the Bauer – Maysenholder-Soeger and the Kaxiras-Pandey Potential Functions. *MATCH Commlin Math. Comput. Chem.,* 54, 29.

Lim. T. C. (2005e). A Functionality Flexible Interatomic Energy Function based on Classical Potentials. *Chem. Phys.*, 320, 54.

Lim. T. C. (2005f). Two body Relationship between the Kaxiras-Pandey and the Murrell-Mottram Potential Functions. *Chinese Journal of Physics*, 43, 43.

Lucha, W., Schoberl, F. F. & Gromis, D. (1991). Bound states of quarks. Physics Rept. 200, 127.

Martin, A. (1988). Collapse of Systems of Relativistic Particles. *Phys. Lett* B214, 561.

Motyka, L. & Zalewski, K. (1996). Spin effects in heavy quarkonia. *Z. Phys*. C69, 343.

Nikiforov, A. F. & Uvarov, V. B. (1988). Special Functions of Mathematical Physics (Birkhauser, Basel).

Oyewumi, K. J. (2005). Analytical Solutions of the Kratzer-Fues Potential in an Arbitrary Number of Dimension. *Foundations of Physics Letters,* 18(1).