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STABILITY OF CERTAIN PROCEDURES AND FIXED POINTS OF NONEXPANSIVE MAPS

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ABSTRACT: We obtain extension of recent results of De la Sen (2010) on (f, T)-stability for the pair satisfying (L, h) property for (δ, k)-weak contractive iteration procedures f. Generalizations of fixed point results concerning altering distance functions are also obtained.

INTRODUCTION

Iteration methods, which are tools for solving real life problems in economics, network analysis, physical sciences and engineering etc, are numerical procedures which compute sequences of gradually accurate iterates to approximate the solutions of classes of problems, Haghi *et al* (2012).

Constructive fixed point theorems are not only mainly concerned with the existence (and, possibly, uniqueness) of fixed points of operators under investigation but provide a method for approximating the desired fixed points. Further, the study of constructive fixed point theorems offers information on the stability of the fixed point iteration procedure. We recall that a fixed point iteration method, intuitively put, is stable if small modifications in the initial data (or any parameter) involved in the iteration process will produce small influence on the value of the approximation. This intuitive notion is formalized in Definition 1 in a complete metric space *(X , d)* for a self map *T* of *X*, some function $f : X \rightarrow X$ which defines the iteration procedure and fixed point set Fix $(T) = \{p \in X \mid p = Tp\}$ of *T*.

Definition 1. (Imoru and Olatinwo(2006))

Let $\{x_n\} \subset X$ *be a sequence generated by an iteration procedure involving T defined by* $x_{n+1} = f(T, x_n), n = 0,1,2,...$ *where* $x_0 \in X$ *is the initial approximation and f some function. Suppose* $\{x_n\}$ *converges to a fixed point p of T. Let* $\{y_n\} \subset X$ *be a sequence and set* $e_n = d(y_{n+1}, f(Ty_n)), n = 0,1,2,...$ *Then the iteration procedure f is said to be T-stable if and only if* $\lim_{n\to\infty} e_n = 0$ *implies that* $\lim_{n\to\infty} y_n = p$.

We must observe that when f (T , x_{n}) = Tx_{n} then we recover the well-known Picard iteration scheme.

Quite a number of researchers have studied stability of various iteration procedures with respect to various types and classes of operators (De la sen (2010), Haghi *et al* (2012), Imoru and Olantinwo (2006), Liu (1990), Osilike (1995), Qin and Rhoades (2008)). Qin and Rhoades (2008) demonstrated that a number of contractive definitions are Picard *T*-stable. Recently, De la sen (2010) investigated *(f , T)*-stability of large-contractions subject to altering-distance functions and extended altering-distance functions. Among the several results proved by De la sen (2010) his Theorem 3.1 is of particular interest to this work. Extension of this result from continuous iteration procedure to *(δ, k)*-contractive iteration procedures *f constitute the main result of this paper*. Our result is significant in considering the more general class of nonexpansive maps instead of contractive requirements on the operators *T*, used by so many contributors in the literature. Further, our result is more significant in that it does not assume existence of fixed points for the class of operators considered which is a complete departure from available contributions and a new direction in the study and investigation of stable procedures. Among the several results proved by De la sen (2012) his Theorem 3.1 is of particular interest in this work.

Theorem 1 (De la sen 2010)

Assume that

1. *(X , d) is a complete metric space,* $f : X \to X$ *is a continuous mapping,* $T : X \to X$ *is a self mapping on X such that the set of fixed points Fix(f , T) of the iteration procedure* $x_{n+1} = f(Tx_n), n = 0,1,2,...$ *is nonempty;*
2. *The (f , T) satisfies (L , h) property; that is* $d\left(f(Tx), p\right) \le Ld\left(f(T, x), x\right) + hd(x, p);$ *for all* $p \in Fix(f, T)$ *with* $0 \le h < 1$ *and* $L \ge 0;$
3. $\lim_{n\to\infty} d\left(f(Tx_n), x_n\right) = 0,$ *for all* $x_0 \in X$ *.*

Then, the iteration procedure $x_{n+1} = f(Tx_n), n = 0,1,2,...$ *is (f, T)-stable and possesses its unique fixed point.*

Definition 2. (*Berinde 2004)*

Let X be a metric space, $\delta \in (0,1)$ *and* $k \ge 0$*, then a mapping* $T : X \to X$ *is called* $(\delta, k)-$ *weak contraction (or a weak contraction) if and only if*

$$d(Tx, Ty) \le \delta d(x, y) + kd(y, Tx), \quad \text{for all } x, y \in X \tag{1.1}$$

In (Berinde2004) Berinde proved the theorem below*:*

Theorem 1.2

Let *(X, d)* be a complete metric space and $T : X \to X$ a *(δ, k)*-weak contraction. Then

(1) Fix $(T) = \{x \in X : Tx = x\} \neq \emptyset$

(2) For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n, n = 0,1,2,...$ converges to some $x^* \in$ Fix (T).

(3) The following estimates

$$d(x_n, x^*) \le \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0,1,2,...$$

$$d(x_n, x^*) \quad \leq \quad \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 0,1,2,...$$

hold, where δ is the constant appearing in eqn. (1.1).

(4) Under the additional condition that there exist $\theta \in (0,1)$ and some $k_1 \geq 0$ such that

$$d(Tx, Ty) \quad \leq \quad \theta\, d(x, y) + k_1.d(x, Tx), \text{ for all } x, y \in X$$

the fixed point *x** is unique and the Picard iteration converges at the rate

$$d(x_n x^*) \leq \theta\, d(x_{n-1}, x^*), \quad n = 1,2,...$$

RESULTS

Theorem 2

Let *(X , d)* be a complete metric space and $T : X \to X$ a continuous self-map of *X*. Suppose $f : X \to X$ is a *(δ, k)*-weak contraction and $\{x_n\}_{n=0}^{\infty}$ is a sequence defined by the procedure $x_{n+1} = f(Tx_n), x_0 \in X$. If:

1. The pair *(f , T)* satisfies the *(L , h)*-property below:

$$d(f(Tx), f(Ty)) \quad \leq \quad hd(x, y) + Ld(Tx, f(Tx)), \quad \text{for all } x \in X; \; p \in \text{Fix } (f)$$

where $L = \max\{L, k\};\; L, k \geq 0$ and $h = \max\{\delta, h\};\; h \in (0,1)$.

2. $d(Tx, f(Ty)) \leq d(x, f(Ty))$

3. *f* commutes with *T* at the fixed point *p* of *f, i.e F(Tp) = Tf(p) for p* $\in$ Fix(f) .

Then *T* and *f* have common unique fixed point.

Proof:

From definition of the procedure $x_{n+1} = f(Tx_n)$ it follows that $x_1 = f(Tx_0)$ and $x_2 = f(Tx_1)$ and we obtain the following, taking into consideration conditions (1) - (3) of Theorem 2.1 and the fact that *f* is a *(δ, k)*-weak contraction:

$$\begin{aligned} d(x_1, x_2) &= d(f(Tx_0), f(Tx_1)) \\ &\leq hd(x_0, x_1) + Ld(Tx_1, f(Tx_0)) \\ &\leq hd(x_0, x_1) + Ld(x_1, f(Tx_0)) \\ &= hd(x_0, x_1) \end{aligned}$$

Also,

$$\begin{aligned} d(x_2,x_3) &= d(f(Tx_1), f(Tx_2)) \\ &\le hd(Tx_1,Tx_2)+Ld(Tx_2 f(Tx_1)) \\ &\le hd(x_1,x_2)+Ld(x_2 f(Tx_1)) \\ &= h^2 d(x_0,x_1). \end{aligned}$$

So, by induction we obtain

$$\begin{aligned} d(x_n,x_{n+1}) &= d(f(Tx_{n-1}), f(Tx_n)) \\ &\le h^n d(x_0,x_1) \end{aligned}$$

Therefore $d(x_{n+k},x_{n+k+1}) \le h^k d(x_{n-1},x_n)$, and $d(x_n,x_{n+1})$ gives

$$d(x_n,x_{n+1}) \le d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{n+l-1},x_{n+l})$$

$$\begin{aligned} &\le h^n d(x_0,x_1)+h^{n+1}d(x_0,x_1)+h^{n+2}d(x_0,x_1)+\cdots+h^{n+l-1}d(x_0,x_1) \\ &= \left[h^n+h^{n+1}+\cdots h^{n+l-1}\right]d(x_0,x_1) \\ &= h^n\left[1+h+\cdots h^{l-1}\right]d(x_0,x_1), l>1 \\ &= \frac{h^n\left[1-h^n\right]}{1-h}d(x_0,x_1). \end{aligned}$$

This shows that $\{x_n\}$ defined by $x_{n+1}=f(Tx_n)$ is a Cauchy sequence, and therefore possesses a limit

$u=f(Tu)$ since *(X,d)* is a complete metric space and *T* is continuous.

We claim that $u=p=f(p)$ since $\{x_n\}$ is a subsequence of another iteration procedure $u_{n+1}=f(u_n)$ with entries restricted to the image of T. So we have $p=f(Tp)=f(p)$ which yields $Tp=Tf(p)$. Using the fact that *f* and *T* commutes at the fixed point p of *f* we obtain $Tp=f(Tp)=p \Rightarrow Tp=p$. Hence, *p* is also a fixed point of *T*. Therefore, *p* is a common unique fixed point of *f* and *T* by the (*L* , *h*)-property. Next, since the fixed point of *f* is equal to the fixed point of *T*, we have that if $\lim e_n=\lim d(y_{n+1}, f(Ty_n))=0$,then $y_n \to p$ as $n\to\infty$ based on the fact that (X,d) is a complete metric space. End of proof.

CONCLUSION

It is easy to reformulate Theorem 3.1 in terms of altering-distance functions as is obtained in De la sen (2010) by replacing the metric by altering-distance functions. It should be observed that uniqueness of fixed point of f follows from (L,h)-property mentioned above.

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