ON THE SEPARATION OF VARIABLES IN GREEN'S FUNCTIONS FOR ELASTIC MATERIAL

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ABSTRACT

In this paper, the method of separation of variables is used in determining the unique solution of Green's function for a homogeneous elastic material. Unlike other methods used by researchers where the strain energy is viewed as a function of dilatational and distortional parts, the method of separation of variables takes a holistic view of the elastic material studied. Due to the advantage of its holistic nature in elastic material treatment, this method facilitates the construction of Green's functions of Laplace's operator for simple boundary value problems. We now illustrate this method with an elastic strip problem as an example.

INTRODUCTION

The method of separation of variables is the most general method for constructing Green's function. In Erbe, (1974), it is shown that the incompressibility constraint allows one to describe standard thermoelastic effect. This approach is supported also by the recent theory proposed by Humphrey and Rajagopol, (1987) where a new theoretical framework for thermoelastic constitutive equations is proposed. The basic idea is to decompose the deformation into two parts: one is due to free uniform heating and the other is due to isothermal mechanical loading. But in our method of separation of variables, the Green's function is determined as a whole without analogous representation into singular and regular parts Humphrey and Rajagopol, (1987). The method of separation of variables makes it possible to construct the Green's functions of the Laplace's operator for the very simple boundary value problems where boundaries are the surfaces of the Cartesian coordinate system (Seremet, 2003). The essence of the method is that the desired function is represented by a set of infinite series with respect to a system of orthonormalised functions with some unknown coefficients. This method is analogous to the regularity assumption that if the strain-energy W is continuously differentiable infinitely many times, then it can be written as an infinite power series in its invariants (Ogden, 1997). Here we assume the function to be for linear differential equations and we employ a well known standard technique for constructing linear differential equations as in (Melnikov, 1995).

The Mathematical Formulation

The Green's function is given as

$$G(x,\xi) = g(x,\xi) + f(x,\xi) \tag{1}$$

where $g(x,\xi)$ and $f(x,\xi)$ are its singular and regular parts respectively and the function:

$$g(x,\xi) = -\frac{1}{2\pi}\ln r(x,\xi) \quad ; \quad f(x,\xi) = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2} \tag{2}$$

are defined for the plane $(-\infty \le x_1, x_2 \le \infty)$ in the Cartesian system of coordinates x_1, x_2.

We consider a given linear representation of a differential equation as

$$L[y(x)] \equiv \sum_{j=0}^{n} q_j(x)\frac{\partial^{n-1}y(x)}{\partial x^{n-1}} = 0, \quad a < x < b \tag{3}$$

with the boundary conditions at the edges of the straight line segment [a, b],

$$M_k(y) \equiv \left[\sum_{j=0}^{n-1} \alpha_{jk} \frac{\partial^j y(a)}{\partial x^j} + \beta_{jk} \frac{\partial^j y(b)}{\partial x^j} \right] = 0, \tag{4}$$

(k = 1, 2,3……..n)

where $q_j(x)$ are the continuous functions for $a < x < b, q_o(x) \neq 0, M_k$ being the linear independent differential operator, α_{ij} and β_{ij} are considered the known constants.

Definitions:

A function $g(x,\xi)$ is said to bea Green's function for the boundary-value problem (3) and (4) if, as a function of the variables x, it possesses the following properties for any fixed $\xi(a,b)$

(i) For both of the semi-intervals $a \leq x < \xi$ and $\xi < x \leq b$, it is a continuous function having continuous derivatives up to the n^{th} order included and satisfies equation (3) within the interval $a < x < \xi$ and $\xi < x < b$

(ii) At $x = \xi$, $g(x,\xi)$ and its derivatives up to the second (n-2)-th order included are continuous functions.

(iii) Its derivatives of the (n-1)-th order is a discontinuous function at

$x = \xi$, provided, $$\frac{\partial^{n-1} g(\xi+0,\xi)}{\partial x^{n-1}} - \frac{\partial^{n-1} g(\xi-0,\xi)}{\partial x^{n-1}} = \frac{1}{q_o(\xi)} \tag{5}$$

(iv) The function $g(x,\xi)$ satisfies the boundary conditions of equation (2).

Thus $$M_k(g) = 0 \quad ,(k = 1,2,3,...n) \tag{6}$$

THEOREM

If the homogenous boundary-value problem given by equations (1) and (2) only has the trivial solution, there exists a unique Green's function for this problem.

Proof:

Let the functions $y_i(x);\ (i = 1,2,3,...,n)$ be the fundamental solution of equation (1). Then in accordance with property (i) of the definition, the Green's function may be represented in the form

$$g(x,\xi) = \sum_{i=1}^{n} a_i^*(\xi) y_i(x), \quad \text{for} \quad a \leq x \leq \xi \tag{7}$$

$$g(x,\xi) = \sum_{i=1}^{n} b_i^*(\xi) y_i(x), \quad \text{for} \quad \xi \leq x \leq b \tag{8}$$

where $a_i^*(\xi)$ *and* $b_i^*(\xi)$ are the unknown functions. Property (ii) leads to a homogeneous set of simultaneous linear algebraic equations of the (n-1)-th order written as follows

$$\sum_{i=1}^{n} C_i(\xi) \frac{\partial^j y(\xi)}{\partial x^j} = 0, \qquad (j = 0,1,2,3,...,n-2) \tag{9}$$

in n unknown functions,

$$C_i(\xi) = b_i^*(\xi) - a_i^*(\xi); \qquad (i = 1,2,3,...,n) \tag{10}$$

Now by property (iii) applied to equations (7) and (8), we have;

$$\sum_{i=1}^{n} (\beta_i'(\xi) - \alpha_i'(\xi)) y_i(x) = \sum C_i(\xi) \frac{\partial^{n-1} y_i(\xi)}{\partial x^{n-1}} = \frac{1}{\rho_0(\xi)} \tag{11}$$

Thus the expressions in equations (9) and (11) constitute a system of n linear algebraic equations in n unknowns $C_i(\xi)$. However, since the functions $y_i(x)$ are the fundamental solutions for equation (1) and consequently, the determinant of the coefficient matrix in the system given by equation (9) to (11) are the Wronskian, which is non-zero.

Hence the system has a unique solution for $c_i(\xi)$. (12)

In order to obtain the unknowns a_i^* and b_i^* in equations(7) and (8), it is necessary to make use of the expressions in equation (10) and the boundary conditions given by equations (2) and (6) of property (iv). In the sequel, $M_k(y)$ should be represented by sums as follows:

$$M_k(y) = A_k(y) + B_k(y); \quad (k = 1,2,3,...n) \tag{13}$$

where $A_k(y)$ *and* $B_k(y)$ are defined as

$$A_k(y) = \sum_{j=0}^{n-1} \alpha_{jk} \frac{\partial^j y(a)}{\partial x^j} \quad , \qquad B_k(y) = \sum_{j=0}^{n-1} \beta_{jk} \frac{\partial^j y(b)}{\partial x^j} \tag{14}$$

Now, by substituting the quantities from equations (7) and (8) into equation (2), we obtain the boundary conditions from equation (6) in the form

$$M_k(g) = \sum_{i=1}^{n} \left(a_i^* A_k(y_i)\right) + \left(b_i^* B_k(y_i)\right) = 0 \tag{15}$$

Motivated by the form of equation (10), then

$$\sum_{i=1}^{n} b_i^* \left[\left(A_k(y_i) + B_k(y_i)\right)\right] = \sum_{i=1}^{n} C_i A_k(y_i) \tag{16}$$

Accounting for representations from equation (13), the final form of the system of linear algebraic equations in $b_i^*(\xi)$ is obtained as

$$\sum_{i=1}^{n} b_i^* M_k(y) = \sum_{i=1}^{n} C_i A_k(y_i), \quad (k = 1,2,3,...,n) \tag{17}$$

Since the form M_k are linearly independent, the determinant of the matrix of the coefficients from equation (17) is non-zero. Consequently, it immediately follows that this system has a unique solution. After determining the values b_i^* from equation (17), the functions $a_i^*(\xi)$ are defined by equation (10), hence the Green's function is symmetrical, i.e. $g(x,\xi) = g(\xi, x)$. Hence the theorem.

Solution of a typical problem for strip

Consider the Green's function $G(x,\xi)$ for Poisson's equation $\nabla_x^2 G(x,\xi) = -\delta(x-\xi)$ for the strip $(-\infty \le x_1 \le \infty, 0 \le x_2 \le a_2)$ (18)

under the conditions

$$\left.\begin{aligned} G = 0\,;\; x_2 = 0 \\ \frac{\partial G}{\partial x_2} = 0\,;\; x_2 = a_2 \end{aligned}\right\} \;, -\infty \le x_1 \le \infty \tag{19}$$

The function $G(x,\xi)$ is restricted at infinity, i.e. $G|_{x=\pm\infty} < \infty$, ; ∇^2 is Laplace's operator, $\delta(x-\xi)$ is the unit Dirac delta

For Strip $(-\infty \le x_1 \le \infty, 0 \le x_2 \le a_2)$ in fig. 1

The general trigonometric series for equation (18) (Seremet, 2003) is given as

$$G = a_0 + \sum_{m=1}^{\infty} a_m \sin v_1 x_2 + \sum_{m=1}^{\infty} b_m \cos v_1 x_2 \tag{20}$$

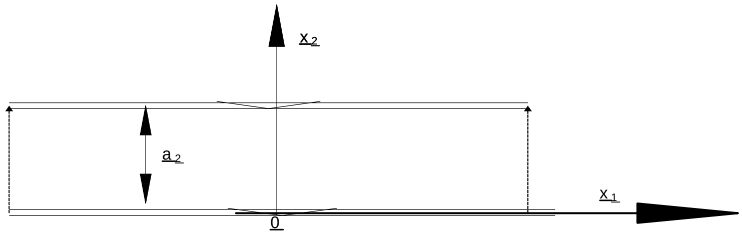

Fig I. strip with boundary straight lines

where a_0, a_m, b_m are the functions of the variable x_1. The boundary conditions of this problem based on the restriction at infinity of x_1 according to (Seremet, 2003) will simplify the general series and reduce it to the form:

$$G = \sum_{m=1}^{\infty} a_m Sin\, v_1 x_2 \qquad v_1 = (2m-1)\frac{\pi}{2a_2} \quad ; \quad m = 1,2,3,... \tag{21}$$

In so doing, the condition $G|_{x_1=\pm\infty} < \infty$ leads to the equivalent conditions

$$a_m|_{x_1=\pm\infty} < \infty \tag{22}$$

By substituting equation (22) for the function G into Poisson's equation in equation (18), we obtain

$$\sum \left(a''_m - v_1{}^2 a_m\right)\sin v_1 x_2 = -\delta(x_1-\xi)\delta(x_2-\xi) \tag{23}$$

Equality $\delta(x-\xi) = \delta(x_1-\xi)\delta(x-\xi)$ being taken into account in the method of separation of variables. We now multiply both parts of equation (24) by $\sin v_2 x_2$, where

$$v_2 = (2s-1)\frac{\pi}{2a_2}, \quad s=1,2,3,... \tag{24}$$

and integrate with respect to the variables x_2; i.e.

$$\int_0^{a_2} \sin(2s-1)\frac{\pi x_2}{2a_2} \sin\ (2m-1)\ \frac{\pi}{2a_2} x_2 dx_2 = \begin{cases} 0; & s \neq m \\ \frac{1}{2}a_2; s = m, v_1 = v_2 \end{cases} \tag{25}$$

$$\int_0^{a_2} \delta\ (x_1-\xi)\ \delta\ (x_2-\xi) \sin v_2\ \ x_2\ \ ax_2 = \delta(x_1-\xi)\sin v_1\xi_2 \quad ; \quad v_1 = v_2 \tag{26}$$

By making use of the property

$$\int_v f(x)a(x-\xi)dv(x) = f(\xi) \tag{27}$$

of the Dirac delta, we obtain from equation (23) the equation

$$\left(a''_m - v_1{}^2\right)\frac{a_2}{2} = -\delta(x_1-\xi)\sin v_1\xi_2 \tag{28}$$

used in determining the function a_m.
Assuming ξ to be a parameter and taking the notation

$$a_m = \frac{2}{a_2}\bar{a}_m \sin v_1\xi_2 \tag{29}$$

we obtain the boundary-value problem for the function $\bar{a}_m$

$$\left(\bar{a}''_m - v_1^2\bar{a}_m\right) = -\delta(x_1-\xi_1); \quad \bar{a}_m|_{x_1=-\infty} < \infty \ ; \ \bar{a}_m|_{x_1=\infty} < \infty \tag{30}$$

The general solution to the equation (30) (Seremet, 2003) is given as

$$\bar{a}_m = \begin{cases} c_1 e^{-v_1 x_1 + c_2 e^{v_1 x}}; x_1 \le \xi \\ k_1 e^{-v_1 x} + k_2 e^{v_1 x_1}; x_1 \le \xi \end{cases} \quad (31)$$

From the conjugality conditions at the point $x_1 = \xi_1$,

$$\bar{a}_m \left(x_1 = \xi - 0\right) = \bar{a}_m \left(x_1 = \xi + 0\right) \quad (32)$$

$$\bar{a}''_m (x_1 = \xi - 0) - \bar{a}_m \left(x_1 = \xi_1 + 0\right) = 1 \quad (33)$$

So we get a set of two simultaneous linear equations

$$(c_1 - k_1)\, e^{-v_1 \xi} + (c_1 - k_2)\, e^{-v_1 \xi} \quad (34)$$

$$v_1 \left[(c_1 - k_1) e^{-v_1 \xi_1} - (c_2 - k_2) e^{v_1 \xi_1} \right] = -1 \quad (35)$$

following from the proof of the theorem [2]

From the boundary conditions at infinity, it follows that

$$\bar{a}_m \left(x_1 = -\infty\right) < \infty \Rightarrow c_1 = 0 \quad (36)$$

$$\bar{a}_m \left(x_1 = \infty\right) < \infty \Rightarrow k_2 = 0 \quad (37)$$

When $c_1 = k_2 = 0$, the solution of the considered set of equations is reduced to the form

$$k_1 = \frac{e^{v_1 \xi}}{2 v_1} \quad ; \quad c_2 = \frac{e^{-v_1 \xi_1}}{2 v_1} \quad ; \quad c_1 = k_2 = 0 \quad (38)$$

Therefore, for the influence function $\bar{a}_m$ in equation (34)**,** the following expression is obtained.

$$\bar{a}_m = \begin{cases} \frac{1}{2 v_1} e^{v_1 (x_1 = \xi_1)} & ; x_1 \le \xi \\ \frac{1}{2 v_1} e^{-v_1 (x_1 = \xi_1)} & ; x_1 \ge \xi \end{cases} \quad (39)$$

With the function $\boldsymbol{a_m = \frac{2}{a_2} \bar{a}_m \sin v_1 \xi_2}$**,** the Green's function $G(x, \xi)$ is written as

$$G(x,\xi) = \begin{cases} G_l(x,\xi) = \frac{2}{a_2} \sum_{m,n=1}^{\infty} \frac{1}{2 v_1} e^{v_1 (x_1 = \xi_1)} \sin v_1 x_2 \sin v_1 \xi_2 & ; x_1 \le \xi_1 \\ G_r(x,\xi) = \frac{2}{a_2} \sum_{m,n=1}^{\infty} \frac{1}{2 v_1} e^{-v_1 (x_1 = \xi_1)} \sin v_1 x_2 \sin v_1 \xi_2 & ; x_1 \ge \xi_1 \end{cases} \quad (40)$$

where $G_l(x, \xi)$ and $G_r(x, \xi)$ denote the expressions for the Green's functions to the left of the action of the unit force when $x_1 \le \xi$ and the right of action, when $x_1 \ge \xi_1$ respectively.

To complete the method of separation of variables, we show that the infinite series in equation (40) can be added together, (Udoh and Sadiku, 2008). In this case, it is sufficient to show that the series

$$\sum_{n=1}^{\infty} \frac{p^{2n-1}}{2n-1} \cos(2n-1)\alpha = \frac{1}{2} \ln \sqrt{\frac{1 + 2p\cos\alpha + p^2}{1 - 2p\cos\alpha + p^2}} \quad , \quad (41)$$

$$p^2 < 1, \quad 0 \le \alpha \le 2\pi \quad or \quad p^2 \le 1, \quad 0 < \alpha < 2\pi$$

Then by making use of the known sum [8]

$$\sum_{n=1}^{\infty} \frac{p^n}{n} \cos n\alpha = -\ln \sqrt{1 - 2p\cos\alpha + p^2}, \quad p^2 < 1, \quad (42)$$

$$0 \le \alpha \le 2\pi \ or \ p^2 \le 1, \quad 0 < \alpha < 2\pi$$

we come to the above statement.

In addition, we can take the sum of the series in the expression for $G(x,\xi)$, using the trigonometric formula $\sin v_1 x_2 \sin v_1 \xi_1 = \frac{1}{2}\left[\cos v_1(x_2-\xi_2)-\cos v_1(x_2+\xi_2)\right]$. (43)

for computation.

So,in equation(40) the symbol n is free and the sums represent ordinary infinite series. Substituting the formula (43) into equation (40) we obtain the expressions

$$G(x,\xi)=G_l(x,\xi)$$
$$=\frac{1}{a_2}\sum_{m=1}^{\infty}\frac{1}{2v_1}e^{v_1(x_1-\xi_1)}[\cos v_1(x_2-\xi_2)-\cos v_1(x_2+\xi_2)],\ x_1\le\xi_1 \tag{44}$$

$$\text{G}(\text{x},\xi)=G_r(x,\xi)=\frac{1}{a_2}\sum_{m=1}^{\infty}\frac{1}{2v_1}e^{-v_1(x_1-\xi_1)}[\cos v_1(x_2-\xi_2)-\cos v_1(x_2+\xi_2)],x_1\ge\xi_1 \tag{45}$$

Taking into account that $v_1=(2m-1)\dfrac{\pi}{2a_2}$, $m=1,2,3,...$ from equation(21)

the expressions (44)-(45) could be written in the form (Udoh, 2011).

$$G(x,\xi)=G_l(x,\xi)=\frac{1}{\pi}$$
$$\sum_{m=1}^{\infty}\frac{1}{(2m-1)}e^{(2m-1)\frac{\pi}{2a_2}(x_1-\xi_1)}[\cos(2m-1)\frac{\pi}{2a_2}(x_2-\xi_2)-\cos(2m-1)\frac{\pi}{2a_2}(x_2+\xi_2)],x_1\le\xi_1 \tag{46}$$

$$G(x,\xi)=G_r(x,\xi)=$$
$$\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)}e^{-(2m-1)\frac{\pi}{2a_2}(x_1-\xi_1)}[\cos(2m-1)\frac{\pi}{2a_2}(x_2-\xi_2)-\cos(2m-1)\frac{\pi}{2a_2}(x_2+\xi_2)],x_1\ge\xi_1 \tag{47}$$

All four series in equations (46), (47) have the form

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{p^{(2n-1)}}{2n-1}\cos(2n-1)\alpha=\frac{1}{2\pi}\left(\sum_{n=1}^{\infty}\frac{p^n}{n}\cos n\alpha-\sum_{n=1}^{\infty}\frac{(-p)^n}{n}\cos n\alpha\right), \tag{48}$$

where p and α take the values

$$p=e^{\frac{\pi}{2a_2}(x_1-\xi_1)};\quad p=e^{-\frac{\pi}{2a_2}(x_1-\xi_1)};\quad \alpha=\frac{\pi}{2a_2}(x_2-\xi_2);\quad \alpha=\frac{\pi}{2a_2}(x_2+\xi_2). \tag{49}$$

where α's are approached from the left and right hand directions respectively.

So, taking into account the known sum [2] in equation (42),we obtain for the sum (48) the expression in the closed form

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{p^{(2n-1)}}{2n-1}\cos(2n-1)\alpha=\frac{1}{2\pi}\ln\sqrt{\frac{1+2p\cos\alpha+p^2}{1-2p\cos\alpha+p^2}};\quad p^2<1,\ . \tag{50}$$
$$0\le\alpha<2\pi\quad or\ p^2\le1,\quad 0<\alpha<2\pi$$

Finally substituting in (50) the values (49) we obtain,

$$G(x,\xi)=G_l(x,\xi)= \tag{51}$$

$$\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)}e^{(2m-1)\frac{\pi}{2a_2}(x_1-\xi_1)}[\cos(2m-1)\frac{\pi}{2a_2}(x_2-\xi_2)-\cos(2m-1)\frac{\pi}{2a_2}(x_2+\xi_2)]=$$

$$\frac{1}{2\pi}\ln\sqrt{\frac{\left[1+2e^{\frac{\pi}{2a_2}(x_1-\xi_1)}\cos\frac{\pi}{2a_2}(x_2-\xi_2)+e^{\frac{\pi}{a_2}(x_1-\xi_1)}\right]\left[1-2e^{\frac{\pi}{2a_2}(x_1-\xi_1)}\cos\frac{\pi}{2a_2}(x_2+\xi_2)+e^{\frac{\pi}{a_2}(x_1-\xi_1)}\right]}{\left[1-2e^{\frac{\pi}{2a_2}(x_1-\xi_1)}\cos\frac{\pi}{2a_2}(x_2-\xi_2)+e^{\frac{\pi}{a_2}(x_1-\xi_1)}\right]\left[1+2e^{\frac{\pi}{2a_2}(x_1-\xi_1)}\cos\frac{\pi}{2a_2}(x_2+\xi_2)+e^{\frac{\pi}{a_2}(x_1-\xi_1)}\right]}}$$

The results (51) and can be written in the form

$$G=\frac{1}{2\pi}\ln\frac{\overline{E}_1\tilde{E}_2}{\tilde{E}_1\overline{E}_2}, \tag{52}$$

where $\frac{\overline{E}_1\tilde{E}_2}{\tilde{E}_1\overline{E}_2}$ represent the top and bottom roots of equation (51).

CONCLUSION

In this paper, a systematic method of separation of variables using Green's function is presented. The method makes use of Green's functions for Poisson's equation and can be extended to Laplaces' equation. For simplicity and without loss of generality, we have used trigonometric functions in constructing the Green's function for the Laplaces' equation. This method can also be used to solve half-strip and rectangular strip problems.

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